Chain Rule Short Cuts

In class we applied the chain rule, step-by-step, to several functions. Here is a short list of examples.

1. Powers of functions
   The rule here is
   \[
   \frac{d}{dx} u(x)^a = au(x)^{a-1}u'(x) \tag{1}
   \]
   So if
   \[
   f(x) = (x + \sin x)^5,
   \]
   then
   \[
   f'(x) = 5(x + \sin x)^4(1 + \cos x).
   \]
The rule (1) is useful when differentiating reciprocals of functions. If \(a = -1\) we get
   \[
   \frac{d}{dx} \frac{1}{u(x)} = \frac{-u'(x)}{u(x)^2}.
   \]
   You could also have derived this using the quotient rule.

2. Exponentials
   For \(a > 0\),
   \[
   \frac{d}{dx} a^{u(x)} = a^{u(x)} u'(x) \ln a, \tag{2}
   \]
   So if
   \[
   g(x) = 3^{x^2-4x},
   \]
   then
   \[
   g'(x) = 3^{x^2-4x} (2x - 4) \ln 3.
   \]

3. The Natural logarithm of a function
   The chain rule in this case says that
   \[
   \frac{d}{dx} \ln u(x) = \frac{1}{u(x)} u'(x) \tag{3}
   \]
   So if
   \[
   f(x) = \ln (\sin x),
   \]
   then
   \[
   f'(x) = \frac{1}{\sin x} \cos x.
   \]
4. Trigonometric functions
We’ll illustrate the chain rule with the cosine function.

\[
\frac{d}{dx} \cos u(x) = -\sin u(x) u'(x) \tag{4}
\]

Thus, if

\[
\psi(x) = \cos (1 + x^3),
\]

then

\[
\psi'(x) = -3x^2 \sin (1 + x^3).
\]

Functions of the form \(\sin u(x)\) and \(\tan u(x)\) are handled similarly.

5. Inverse trigonometric functions
We’ll use the \(\arctan\) function. The chain rule tells us that

\[
\frac{d}{dx} \arctan u(x) = \frac{1}{1 + u(x)^2} u'(x). \tag{5}
\]

So if

\[
\varphi(x) = \arctan (x + \ln x),
\]

then

\[
\varphi'(x) = \frac{1}{1 + (x + \ln x)^2} \left(1 + \frac{1}{x}\right).
\]

Functions of the form \(\arcsin u(x)\) and \(\arccos u(x)\) are handled similarly.

Bear in mind that you might need to apply the chain rule as well as the product and quotient rules to take a derivative. You might also need to apply the chain rule more than once. For example,

\[
\frac{d}{dx} \sin (\ln (x - 2x^2)) = \cos (\ln (x - 2x^2)) \frac{1}{x - 2x^2} (1 - 4x).
\]