Math 843 Final Exam. Do four problems. Show your work.

1. The initial value problem for the heat equation with periodic boundary conditions is

\[ (P_1) \begin{cases} u_t - u_{xx} = 0, & \text{for } 0 \leq x \leq 1, \ t > 0, \\ u(x,0) = f(x), \\ u(0,t) - u(1,t) = 0, \\ u_x(0,t) - u_x(1,t) = 0. \end{cases} \]

5 a. Set \( u(x,t) = X(x)T(t) \). Derive an ODE for \( T \) and an eigenvalue problem for \( X \).

5 b. Show that the eigenvalue problem is self-adjoint.

5 c. Find the eigenvalues. Describe the eigenspaces. Find an orthonormal basis for each eigenspace.

5 d. Write the solution in the form

\[ u(x,t) = \int_0^1 G(x,\xi,t) f(\xi) \ d\xi. \]  

Give \( G \) explicitly.

2. Let \( Q \) be a blob in \( \mathbb{R}^3 \) with a smooth, closed boundary \( \partial Q \). Let \( \nu \) be the outer unit normal to \( \partial Q \).

10 a. Derive the identity

\[ \int_Q w \Delta w \ dx = \int_{\partial Q} w \nabla w \cdot \nu \ dS - \int_Q |\nabla w|^2 \ dx. \]  

(2)

10 b. Use (2) to prove the unicity of the solution to the Dirichlet problem

\[ (P_2) \begin{cases} \Delta u(x) = 0 & \text{for } x \in Q, \\ u(x) = f(x), & \text{for } x \in \partial Q. \end{cases} \]

3. A thin, elastic membrane occupies a region \( Q \subset \mathbb{R}^2 \) that is bounded by the smooth, closed curve \( \partial Q \). The membrane is clamped along \( \partial Q \). Small-amplitude vibrations on \( Q \) are modeled by an initial-boundary value problem for the wave equation

\[ (P_3) \begin{cases} u_{tt} - \Delta u = 0 & \text{for } x \in Q, \ t > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x), \\ u|_{\partial Q} = 0. \end{cases} \]

10 a. Derive the energy identity

\[ \frac{1}{2} \int_Q [u_t(x,t)^2 + |\nabla u(x,t)|^2] \ dx = \text{Constant}. \]  

(3)

10 b. Use (3) to establish the unicity of the solution to \( (P_3) \).
4. For $D > 0$, $t > 0$ and $x \in \mathbb{R}^n$, let

$$G(x, t) = (4\pi Dt)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4Dt}\right).$$  \hspace{1cm} (4)

[10] a. Compute the Fourier transform $\hat{G}(\xi, t)$.

[10] b. Solve the initial value problem for the heat equation,

$$\begin{cases}
  u_t - D\Delta u = 0 & \text{for } x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) = f(x),
\end{cases}$$

[20] 5. Consider the initial value problem for the linear, inhomogeneous Schrödinger equation,

$$\begin{cases}
  u_t = iu_{xx} + g(x, t) & \text{for } x \in \mathbb{R}, t > 0, \\
  u(x, 0) = f(x),
\end{cases}$$

where $f$ is smooth and decays rapidly as $|x| \to \infty$. Use the Fourier transform to derive a solution representation

$$u(x, t) = \int_{-\infty}^{\infty} K(x, y, t)f(y) \, dy + \int_0^t \int_{-\infty}^{\infty} K(x, y, t-s)g(y, s) \, dy \, ds. \hspace{1cm} (5)$$

Give $K$ explicitly. (Hint: Just pretend that $i$ is a real diffusion coefficient. This raises doubts about the convergence of the integrals in (5). Ignore the doubts and proceed by the method of wishful thinking.)

[20] 6. Use the method of characteristics to solve the initial value problem

$$\begin{cases}
  u_t + 2tu_x - e^{-u} = 0 & \text{for } x \in \mathbb{R}, t > 0, \\
  u(x, 0) = x.
\end{cases}$$

[20] 7. Consider the initial value problem

$$\begin{cases}
  u_t + u^3u_x = 0 & \text{for } x \in \mathbb{R}, t > 0, \\
  u(x, 0) = g(x),
\end{cases}$$

where

$$g(x) = \begin{cases}
  1 & \text{for } x \leq 0, \\
  1 - x & \text{for } 0 < x \leq 1, \\
  0 & \text{for } x > 1.
\end{cases}$$

Does the solution undergo gradient blowup at some finite time $t_b$? If so, give $t_b$. 
8. Consider the initial value problem

\[(P_8) \begin{cases} u_t + uu_x = 0 & \text{for } x \in \mathbb{R}, \ t > 0, \\ u(x, 0) = f(x), \end{cases}\]

where

\[f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^2 & \text{for } x > 0. \end{cases}\]

a. Does the solution undergo gradient blowup at some finite time \(t_b\)? If so, give \(t_b\).

b. Solve \((P_8)\).

9. Let \(Q(t)\) be a material volume in a fluid flowing with velocity field \(v\). Derive the convection theorem,

\[
\frac{d}{dt} \int_{Q(t)} g(x, t) \, dx = \int_{Q(t)} \left[ Dg \frac{Dt}{Dt} + g \nabla \cdot v \right] \, dx.
\]

You may assume the Euler expansion formula for the Jacobian \(J\).

10. Let \(w(x, y) = (u(x, y), -v(x, y))\) be the velocity field of a steady, two-dimensional, \textit{incompressible} flow. Suppose, moreover, that the flow is irrotational, so that \(w(x, y) = \nabla \varphi(x, y)\) for some potential function \(\varphi(x, y)\). Show that

\[
ux = vy, \quad \text{and} \quad uy = -vx.
\]

(Note that these are the just the Cauchy-Riemann equations from complex variable theory. (7) shows (subject to the continuity of the partial derivatives) that \(u(x, y) - iv(x, y)\) is an analytic function. Thus many problems in two-dimensional, steady, incompressible flow can be solved by complex variable methods, in particular by conformal mapping.)