Math 843 Final Exam. Do four problems. Show your work.

1. The initial value problem for the heat equation with periodic boundary conditions is

$$(P_1)\begin{cases} u_t - u_{xx} = 0, & \text{for } 0 \le x \le 1,\ t > 0, \\ u(x,0) = f(x), & \\ u(0,t) - u(1,t) = 0, & \\ u_x(0,t) - u_x(1,t) = 0. & \end{cases}$$

[5] **a.** Set $u(x,t) = X(x)T(t)$. Derive an ODE for T and an eigenvalue problem for X.

[5] **b.** Show that the eigenvalue problem is self-adjoint.

[5] **c.** Find the eigenvalues. Describe the eigenspaces. Find an orthonormal basis for each eigenspace.

[5] **d.** Write the solution in the form

$$u(x,t) = \int_0^1 G(x,\xi,t) f(\xi)\, d\xi. \tag{1}$$

Give G explicitly.

2. Let Q be a blob in $\mathbf{R}^3$ with a smooth, closed boundary ∂Q. Let ν be the outer unit normal to ∂Q.

[10] **a.** Derive the identity

$$\int_Q w\Delta w\, dx = \int_{\partial Q} w \nabla w \cdot \nu\, dS - \int_Q |\nabla w|^2\, dx. \tag{2}$$

[10] **b.** Use (2) to prove the unicity of the solution to the Dirichlet problem

$$(P_2)\begin{cases} \Delta u(x) = 0 & \text{for } x \in Q, \\ u(x) = f(x), & \text{for } x \in \partial Q. \end{cases}$$

3. A thin, elastic membrane occupies a region $Q \subset \mathbf{R}^2$ that is bounded by the smooth, closed curve ∂Q. The membrane is clamped along ∂Q. Small-amplitude vibrations on Q are modeled by an initial-boundary value problem for the wave equation

$$(P_3)\begin{cases} u_{tt} - \Delta u = 0 & \text{for } x \in Q,\ t > 0, \\ u(x,0) = f(x), & \\ u_t(x,0) = g(x), & \\ u\big|_{\partial Q} = 0. & \end{cases}$$

[10] **a.** Derive the energy identity

$$\frac{1}{2}\int_Q \left[u_t(x,t)^2 + |\nabla u(x,t)|^2 \right] dx = \text{Constant}. \tag{3}$$

[10] **b.** Use (3) to establish the unicity of the solution to (P_3).

4. For $D > 0$, $t > 0$ and $x \in \mathbf{R}^n$, let

$$G(x,t) = (4\pi D t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4Dt}\right). \tag{4}$$

[10] **a**. Compute the Fourier transform $\hat{G}(\xi, t)$.

[10] **b**. Solve the initial value problem for the heat equation,

$$(P_4) \begin{cases} u_t - D\Delta u = 0 & \text{for } x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = f(x), \end{cases}$$

[20] **5**. Consider the initial value problem for the linear, inhomogeneous Schrödinger equation,

$$(P_5) \begin{cases} u_t = iu_{xx} + g(x,t) & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = f(x), \end{cases}$$

where is f is smooth and decays rapidly as $|x| \to \infty$. Use the Fourier transform to derive a solution representation

$$u(x,t) = \int_{-\infty}^{\infty} K(x,y,t) f(y)\, dy + \int_0^t \int_{-\infty}^{\infty} K(x,y,t-s) g(y,s)\, dy\, ds. \tag{5}$$

Give K explicitly. (Hint: Just pretend that i is a real diffusion coefficient. This raises doubts about the convergence of the integrals in (5). Ignore the doubts and proceed by the method of wishful thinking.)

[20] **6**. Use the method of characteristics to solve the initial value problem

$$(P_6) \begin{cases} u_t + 2tu_x - e^{-u} = 0 & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = x. \end{cases}$$

[20] **7**. Consider the initial value problem

$$(P_7) \begin{cases} u_t + u^3 u_x = 0 & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = g(x), \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 - x & \text{for } 0 < x \le 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Does the solution undergo gradient blowup at some finite time t_b? If so, give t_b.

[20] **8**. Consider the initial value problem

$$(P_8)\begin{cases} u_t + uu_x = 0 & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = f(x), \end{cases}$$

where

$$f(x) = \begin{cases} 0 & \text{for } x \le 0, \\ x^2 & \text{for } x > 0. \end{cases}$$

[10] **a**. Does the solution undergo gradient blowup at some finite time t_b? If so, give t_b.

[10] **b**. Solve (P_8).

[20] **9**. Let $Q(t)$ be a material volume in a fluid flowing with velocity field v. Derive the convection theorem,

$$\frac{d}{dt}\int_{Q(t)} g(x,t)\,dx = \int_{Q(t)} \left[\frac{Dg}{Dt} + g\nabla\cdot v\right] dx. \tag{6}$$

You may assume the Euler expansion formula for the Jacobian J.

[20] **10**. Let $w(x,y) = (u(x,y), -v(x,y))$ be the velocity field of a steady, two-dimensional, *incompressible* flow. Suppose, moreover, that the flow is irrotational, so that $w(x,y) = \nabla\varphi(x,y)$ for some potential function $\varphi(x,y)$. Show that

$$u_x = v_y, \quad \text{and} \quad u_y = -v_x. \tag{7}$$

(Note that these are the just the Cauchy-Riemann equations from complex variable theory. (7) shows (subject to the continuity of the partial derivatives) that $u(x,y) - iv(x,y)$ is an analytic function. Thus many problems in two-dimensional, steady, incompressible flow can be solved by complex variable methods, in particular by conformal mapping.