Math 843 Exam 2

Do three problems.

Show your work.

1. Use the Fourier transform to derive the D'Alembert representation

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz.$$

for the solution to the initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x). \end{cases}$$

2. Solve the initial value problem for the inhomogeneous heat equation,

$$\begin{cases} u_t - Du_{xx} = g(x,t), & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = 0. \end{cases}$$

Write the solution in the form

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,y,t-s)g(y,s)\,dyds,$$

and give G explicitly.

3. Use the Fourier transform to find a bounded solution to the Dirichlet problem on the upper half-plane,

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for } x \in \mathbf{R} \text{ and } y > 0, \\ u(x,0) = f(x). \end{cases}$$

Write the solution in the form

$$u(x,y) = \int_{-\infty}^{\infty} K(x,y)f(y)\,dy,$$

and give K explicitly.

4. Solve the initial value problem

$$\begin{cases} u_t + xu_x = u^4, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2. \end{cases}$$

5. Consider the initial value problem

$$\begin{cases} u_t + u^3 u_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x), \end{cases}$$

where

$$f(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ (1-x)^{\frac{1}{3}} & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Sketch the characterisic diagram. Solve for $u(x,t)$. Does the solution become discontinuous in finite time?

6. Consider the initial value problem

$$\begin{cases} u_t + a(u)u_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x). \end{cases}$$

Show that the breaking time t_b is the smallest time t at which

$$1 + ta'(f(\xi))f'(\xi) = 0,$$

for some point ξ.