Well-Posed Problems

1. The problems discussed so far follow a pattern: Elliptic PDEs are coupled with boundary conditions, while hyperbolic and parabolic equations get initial-boundary and pure initial conditions. This is part of a more general pattern. Certain types of partial differential equations go naturally with certain side conditions.

2. According to Hadamard, a problem is well-posed (or correctly-set) if

a. it has a solution,

b. the solution is unique,

c. the solution depends continuously on data and parameters.

The meaning of (a) is clear. When we call a solution "unique," we sometimes mean "unique within a certain class of functions." For example, a problem might have several solutions, only one of which is bounded. We'd say that the solution was unique in the space of bounded functions. And a solution depends continuously on data and parameters if "small" changes in initial or boundary functions (in appropriate norms) and in parameter values result in "small" changes in the solution (in some appropriate norm).

3. The notion of a well-posed problem is important in applied math. If you were using an initial-boundary value problem (P) to make predictions about some physical process, you'd obviously like (P) to have solution. You'd also want to be sure of the solution's unicity. And if the solution depends continuously on data and parameters, you don't have to worry about small errors in measurement producing large errors in your predictions.

4. Though the classical theory of partial differential equations deals almost completely with the well-posed, ill-posed problems can be mathematically and scientifically interesting.

5. Let $\| \;\|$ be a norm on a linear space V. We'll write the norm of $u(x,t)$ as a function of x at a fixed time t as $\|u(t)\|$. For example, the $L^1(B)$ norm of u at time t is

$$\|u(t)\| = \int_B |u(x,t)|\,dx. \tag{1}$$

6. **Example**: Let $0 < \varepsilon \ll 1$, and $\| \;\|$ be the maximum norm on $C(\mathbf{R})$. A solution to

$$\begin{cases} u_{tt} + u_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = 0, \\ u_t(x,0) = 0, \end{cases}$$

is $u(x,t) \equiv 0$. A solution to

$$\begin{cases} v_{tt} + v_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ v(x,0) = 0, \\ v_t(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right), \end{cases}$$

is

$$v(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right) \sinh\left(\frac{t}{\varepsilon}\right). \tag{2}$$

Note that u and v satisfy the same equation, have the same initial value, and that their time derivatives are initially close:

$$\|v_t(0) - u_t(0)\| = \varepsilon.$$

Nevertheless,

$$\|v(t) - u(t)\| = \varepsilon^2 \left|\sinh\left(\frac{t}{\varepsilon}\right)\right|, \tag{3}$$

which is exponentially large. Thus a very small change in the initial data results in a large change in the solution for positive time. It isn't hard to explain this. Though the solution v is time-dependent, it satisfies an elliptic equation (Laplace's) that goes naturally with time-independent problems. The pairing of the elliptic equation with initial conditions led to an ill-posed problem.

7. Suppose we change the Laplacian to the (hyperbolic) wave operator. Our problems are now

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = 0, \\ u_t(x,0) = 0, \end{cases}$$

with solution $u(x,t) \equiv 0$, and

$$\begin{cases} v_{tt} - v_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ v(x,0) = 0, \\ v_t(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right), \end{cases}$$

with

$$v(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right) \sin\left(\frac{t}{\varepsilon}\right). \tag{4}$$

In this case,

$$\|v(t) - u(t)\| = \varepsilon^2 \left|\sin\left(\frac{t}{\varepsilon}\right)\right| \le \varepsilon^2. \tag{5}$$

Thus the small change in the initial data leads a small change in the solution at any positive time. One expects this, since the initial value problem is well-posed for the wave equation.

8. Establishing (a) can be quite difficult. We'll do this only with simple problems for which one can write down a solution. Proving (b) and (c) is usually easier, especially for linear problems.

9. **Example**: Consider intitial-boundary value problem for the heat equation with source $f(x,t)$:

$$(P)\begin{cases} u_t - k\Delta u = f(x,t), & \text{for } x \in B \text{ and } t > 0, \\ u(x,0) = h(x), \\ u\big|_{\tilde{Q}} = g(x,t). \end{cases}$$

We take B to be a bounded region in $\mathbf{R}^n$ enclosed by a "nice" surface Q, and f, g and h to be continuous functions of their arguments on appropriate domains. Let $\|\ \|_2$ be the $L^2(B)$ norm. If u and v smooth solutions, then $w = u - v$ satisfies

$$\begin{cases} w_t - k\Delta w = 0, & \text{for } x \in B \text{ and } t > 0, \\ w(x,0) = 0, \\ w\big|_{\tilde{Q}} = 0. \end{cases}$$

Multiply the equation by w, integrate by parts and use the boundary data to show that

$$\frac{d}{dt}\|w(t)\|_2^2 = -2k\int_B |\nabla w|\,dx \le 0.$$

Hence

$$0 \le \|w(t)\|_2^2 \le \|w(0)\|_2^2 = 0.$$

So for each fixed $t \ge 0$,

$$\|u(t) - v(t)\|_2 = 0.$$

It follows that $u(x,t) = v(x,t)$ for $x \in B$ and $t \ge 0$. Thus there is a unique smooth solution to (P).

10. In certain problems, expressions like $\|\psi(t)\|_2^2$ represent energy. Methods like the one used above are for this reason called energy methods. Their use is not restricted to unicity arguments for linear, parabolic problems. They are ubiquitous in PDE, appearing in all parts of well-posedness proofs for all sorts of problems—linear and nonlinear, parabolic, hyperbolic, elliptic and mixed.