Wave Propagation 5: The Soliton Solution to the KdV Equation

1. Let $\kappa > 0$. The *Korteweg-deVries equation* is

$$v_t + vv_x + \kappa v_{xxx} = 0. \tag{1}$$

It is usually referred to as the *KdV equation*. Note that it is a third-order, nonlinear partial differential equation. As with the Burgers equation, we seek a traveling wave solution, i.e. one of the form

$$v(x,t) = g(z), \tag{2}$$

where

$$z = x - ct.$$

Plug $g(z)$ into (1) to obtain the ODE

$$-cg' + \frac{1}{2}\frac{d}{dz}g^2 + \kappa g''' = 0.$$

Integrate once to get

$$-cg + \frac{1}{2}g^2 + \kappa g'' = A,$$

where A is a constant of integration. Multiply through by g':

$$-cgg' + \frac{1}{2}g^2 g' + \kappa g' g'' = Ag',$$

and integrate again:

$$-\frac{c}{2}g^2 + \frac{1}{6}g^3 + \frac{1}{2}\kappa g'^2 = Ag + B, \tag{3}$$

where B is another constant of integration. Now solve for g'.

$$\frac{dg}{dz} = \pm\sqrt{\frac{1}{3\kappa}}\,P(g)^{\frac{1}{2}}, \tag{4}$$

where P is the cubic polynomial

$$P(g) = -g^3 + 3cg^2 + 6Ag + 6B. \tag{5}$$

2. Take the positive root in (4) and suppose that P has two real distinct zeros: α, of multiplicity one and β of multiplicity two, with $\beta < \alpha$. P has the

$$P(g) = (g - \beta)^2(\alpha - g). \tag{6}$$

So for $\beta < g < \alpha$, equation (4) can be written

$$\frac{dz}{\sqrt{3\kappa}} = \frac{dg}{(g-\beta)(\alpha-g)^{\frac{1}{2}}}. \tag{7}$$

3. After some tedious but not particularly difficult integration we arrive at the solution

$$g(z) = \beta + (\alpha-\beta)\,\text{sech}^2\left(\sqrt{\frac{\alpha-\beta}{12\kappa}}\,z\right). \tag{8}$$

By comparing expressions (5) and (6) for $P(g)$, we see that the wave speed is

$$c = \frac{\alpha+2\beta}{3}. \tag{9}$$

Thus the traveling wave solution is

$$v(x,t) = \beta + (\alpha-\beta)\,\text{sech}^2\left(\sqrt{\frac{\alpha-\beta}{12\kappa}}\left(x - \left[\beta + \frac{\alpha-\beta}{3}\right]t\right)\right). \tag{10}$$

For various reasons, (10) is called a *soliton.* It is a special type of *solitary wave.* The terms "soliton" and "solitary wave" are often used interchangeably, though they shouldn't be. A soliton has properties that set it apart from the general class of solitary waves.