Wave Propagation 4: Traveling Waves

1. The simplest, nonlinear conservation law is

$$u_t + uu_x = 0. \tag{1}$$

This is the inviscid Burgers equation. The flux for (1) is

$$J(u) = \frac{u^2}{2}. \tag{2}$$

We introduce diffusion by taking

$$J(u) = \frac{u^2}{2} - \nu u_x, \tag{3}$$

where $\nu > 0$. The corresponding conservation law is the viscous Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0. \tag{4}$$

You might wonder why (4) is called the "viscous" rather than the "diffusive" Burgers equation. The answer is that in the simplest model of fluid flow, viscosity arises from diffusion between adjacent fluid elements. Thus the term νu_{xx} can be thought of as representing diffusion or viscosity.

2. On physical grounds alone, one would expect viscosity to have a smoothing effect. To check this, consider the initial value problem for the diffusion equation with large, localized initial value modeled by the delta function:

$$(P_0)\begin{cases} u_t - \nu u_{xx} = 0, & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = \delta(x). \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{x^2}{4\nu t}},$$

which is infinitely differentiable for positive t. The diffusion has "smeared out" the initial distribution.

3. Advection, on the other hand, does not smooth. The initial value problem for the simplest, linear, advection equation is

$$(P_1)\begin{cases} w_t + cw_x = 0, & \text{for } x \in \mathbf{R},\ t > 0, \\ w(x,0) = \delta(x). \end{cases}$$

The solution is

$$w(x,t) = \delta(x - ct),$$

which is no smoother than its initial value. Indeed, we've seen that nonlinear advection equations admit shock solutions that are less smooth than their initial data.

4. We will use the viscous Burgers equation to study the combined effects of diffusion and nonlinear advection. Since there is no (obvious) way to solve the initial value problem for (4), we'll employ the common strategy of seeking "special" solutions—in this case, traveling waves, the simplest of which are functions of the form

$$u(x,t) = g(x - ct). \tag{5}$$

Let

$$z = x - ct.$$

Then $u_x = g'(z)$ and $u_t = -cg'(z)$, and the viscous Burgers equation becomes

$$-cg' + gg' - \nu g'' = 0.$$

Integrate once to get

$$-cg + \frac{1}{2}g^2 - \nu g' = A,$$

where A is a constant of integration. Thus,

$$\begin{aligned} g' &= \frac{1}{2\nu}\left(g^2 - 2cg - 2A\right) \\ &= \frac{1}{2\nu}(g - g_1)(g - g_2), \end{aligned} \tag{6}$$

where

$$g_1 = c - \sqrt{c^2 + 2A} \quad \text{and} \quad g_2 = c + \sqrt{c^2 + 2A}. \tag{7}$$

Assume that $c^2 + 2A > 0$, so that $g_2 > g_1$. We can rewrite (6) as

$$\frac{dg}{(g - g_1)(g - g_2)} = \frac{dz}{2\nu}.$$

By partial fractions, this is

$$\frac{1}{g_2 - g_1} \ln\left|\frac{g - g_2}{g - g_1}\right| = \frac{z}{2\nu} + B, \tag{8}$$

where B is a constant. If we take $B = 0$ then (8) is

$$\frac{1}{g_2 - g_1} \ln\left(\frac{g_2 - g}{g - g_1}\right) = \frac{z}{2\nu}, \tag{9}$$

for $g_1 < g < g_2$. We set

$$K = \frac{g_2 - g_1}{2\nu} > 0,$$

and solve (9) for

$$g(z) = \frac{g_2 + g_1 e^{Kz}}{1 + e^{Kz}}. \tag{10}$$

Thus g is a smooth function that decreases monotonically from $g(-\infty) = g_2$ to $g(\infty) = g_1$, with an inflection point at $z = 0$. (We say that g *connects* the states g_2 and g_1.)

5. The function

$$u(x,t) = g(x - ct) = \frac{g_2 + g_1 e^{K(x-ct)}}{1 + e^{K(x-ct)}}, \tag{11}$$

is called the *shock structure solution* to the viscous Burgers equation. It is an example of a *traveling wave* or *wavefront* solution. It clearly represents a wavefront moving with velocity c. By (7),

$$c = \frac{g_2 + g_1}{2}. \tag{12}$$

6. Note that

$$\lim_{\nu \to 0^+} u(x,t) = \begin{cases} g_2 & \text{for } x < ct, \\ g_1 & \text{for } x > ct. \end{cases}$$

Thus, as the viscosity tends to zero, its smoothing effect is lost, and solution becomes discontinuous.