Wave Propagation 3

1. In the last set of notes, we saw that the solution u to

$$(P_0)\begin{cases} u_t + J(u)_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x), \end{cases}$$

does not generally exist as a continuous function for all time. There can be a breaking time t_b at which a shock, or discontinuity forms. We will use a "jump condition" to extend the solution in a physically reasonable way beyond the breaking time. Since the solution loses regularity (smoothness) at t_b, we will replace the differential form

$$u_t + J(u)_x = 0, \tag{1}$$

of the conservation law with the integral form:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + J(u(b,t)) - J(u(a,t)) = 0, \tag{2}$$

for all $a < b$ and $t > 0$. Thus at any time t, over any interval $[a,b]$, mass is conserved, but we do not require continuity of u.

2. The Jump Condition: Suppose that a shock curve $x = s(t)$ divides some domain D into regions D_0 to the left of the shock and D_1 to the right. Suppose that u is bounded and satisfies (2) on D. Let

$$u(x,t) = \begin{cases} u_0(x,t) & \text{for } (x,t) \in D_0, \\ u_1(x,t) & \text{for } (x,t) \in D_1, \end{cases}$$

where u_0 and u_1 are C^1 on their respective domains. For any function $g : \mathbf{R} \mapsto \mathbf{R}$, set

$$[g(u(t))] = g(u_0(s(t),t)) - g(u_1(s(t),t),$$

be the jump in the value of $g(u)$ across the shock at time t. Fix t and let $a < s(t) < b$. By (2) and the Leibniz rule

$$\begin{aligned} J(u_0(a,t)) - J(u_1(b,t)) &= \frac{d}{dt}\int_a^b u\,dx \\ &= \frac{d}{dt}\int_a^{s(t)} u_0\,dx + \frac{d}{dt}\int_{s(t)}^b u_1\,dx \\ &= [u_0(s(t),t) - u_1(s(t),t)]\dot{s}(t) + \int_a^{s(t)} u_{0_t}\,dx + \int_{s(t)}^b u_{1_t}\,dx \\ &= [u(t)]\dot{s}(t) + \int_a^{s(t)} u_{0_t}\,dx + \int_{s(t)}^b u_{1_t}\,dx. \end{aligned} \tag{3}$$

Let $a \uparrow s(t)$ and $b \downarrow s(t)$. The left-hand side of (3) goes to $[J(u(t))]$, and since u_{0_t} and u_{1_t} are continuous on their respective domains, the integrals on the right-hand side vanish. We're left with the jump condition

$$[J(u(t))] = [u(t)]\dot{s}(t). \tag{4}$$

3. **Example**: Consider the initial value problem

$$(P_0) \begin{cases} u_t + uu_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = g(x), \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - x & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$$

In this problem, the flux density is

$$J(u) = \frac{u^2}{2}. \tag{5}$$

As we saw in the previous notes the breaking time for this problem is $t_b = 1$, and the breaking point, $x_b = 1$. For $t < 1$,

$$u(x,t) = \begin{cases} 1 & \text{for } x \leq t, \\ \frac{1-x}{1-t} & \text{for } t < x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Let D be the half-plane $t > 1$. Assume that a shock curve $x = s(t)$ emanating from $(x_b, t_b) = (1,1)$ splits D into left and right regions D_0 and D_1 respectively. Given the solution u before $t_b = 1$, it is natural to stipulate that

$$u(x,t) = \begin{cases} 1 & \text{for } (x,t) \in D_0, \\ 0 & \text{for } (x,t) \in D_1. \end{cases}$$

So by the jump condition (4) and (5),

$$\dot{s}(t) = \frac{[J(u(t))]}{[u(t)]} = \frac{1}{2}. \tag{6}$$

The shock curve is therefore,

$$x = s(t) = \frac{1}{2}(t+1). \tag{7}$$

We thus extend the solution across $t_b = 1$ by setting

$$u(x,t) = \begin{cases} 1 & \text{for } t \geq 1 \text{ and } x < \frac{1}{2}(t+1), \\ 0 & \text{for } t \geq 1 \text{ and } x > \frac{1}{2}(t+1). \end{cases}$$