

Wave Propagation 2

1. We began the study of the first-order quasilinear equation as a generalization of the first-order conservation law,

$$u_t + J(u)_x = 0, \quad (1)$$

or

$$u_t + a(u)u_x = 0, \quad (2)$$

where $a(u) = J'(u)$. The initial value problem is

$$(P_0) \begin{cases} u_t + a(u)u_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x, 0) = f(x), \end{cases}$$

and the characteristic system

$$(i) \begin{cases} \dot{x} = a(z), \\ x(\xi, 0) = \xi, \end{cases} \quad (ii) \begin{cases} \dot{t} = 1, \\ t(\xi, 0) = 0, \end{cases} \quad (iii) \begin{cases} \dot{z} = 0, \\ z(\xi, 0) = f(\xi). \end{cases} \quad (3)$$

The solution to (iii) is

$$z = f(\xi). \quad (4)$$

Plug this into (i) and solve for

$$x = a(f(\xi))s + \xi. \quad (5)$$

The solution to (ii) is

$$t = s. \quad (6)$$

And hence,

$$x = a(f(\xi))t + \xi. \quad (7)$$

In principle, we solve (7) for

$$\xi = \xi(x, t), \quad (8)$$

and thus obtain the solution

$$u(x, t) = z(\xi(x, t)) = f(\xi(x, t)). \quad (9)$$

2. For problem (P_0) , note that

a. The characteristics are the straight lines in the xt -plane given by (7). Thus the characteristic through $(\xi, 0)$ has slope $a(f(\xi))^{-1}$.

b. Differentiate the solution u along the characteristic (7):

$$\begin{aligned} \frac{d}{dt}u(x, t) &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial t} \\ &= a(f(\xi))u_x + u_t \\ &= a(u)u_x + u_t \\ &= 0. \end{aligned}$$

So along the characteristic line $x(\xi, t) = a(f(\xi))t + \xi$, the solution u has constant value

$$u(x(\xi, t), t) \equiv u(x(\xi, 0), 0) = f(\xi). \quad (10)$$

3. Example: Let

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } 0 < x \leq 1, \\ 1 & \text{for } x > 1, \end{cases}$$

and

$$J(u) = \frac{u^2}{2}. \quad (11)$$

Thus $a(u) = J'(u) = u$, and the initial value problem (P_0) becomes

$$(P_1) \begin{cases} u_t + uu_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x, 0) = f(x). \end{cases}$$

Thus,

a. For $\xi \leq 0$ the characteristics are

$$\begin{aligned} x &= a(f(\xi))t + \xi \\ &= f(\xi)t + \xi \\ &= \xi. \end{aligned}$$

These are vertical lines in the xt -plane. Since $u \equiv f(\xi) = 0$ along each one,

$$u(x, t) \equiv 0 \quad \text{for } x \leq 0.$$

b. For $0 < \xi \leq 1$ the characteristics are

$$\begin{aligned} x &= a(f(\xi))t + \xi \\ &= \xi t + \xi \\ &= \xi(1 + t). \end{aligned} \quad (12)$$

Thus the characteristic emanating from $\xi = 0$ is vertical, and the one from $\xi = 1$ has equation $x = t + 1$. In the zone bounded by these two lines we have (by (12)),

$$\xi = \frac{x}{1 + t}.$$

And thus,

$$u(x, t) = f(\xi(x, t)) = \frac{x}{1 + t} \quad \text{for } 0 < x \leq 1 + t. \quad (13)$$

c. For $\xi > 1$ the characteristics are

$$\begin{aligned} x &= a(f(\xi))t + \xi \\ &= f(\xi)t + \xi \\ &= t + \xi. \end{aligned}$$

These are lines of slope 1 in the xt -plane. Since $u \equiv f(\xi) = 1$ along each one,

$$u(x, t) \equiv 0 \quad \text{for } x > 1 + t. \quad (14)$$

d. To sum up, the solution to (P_1) is

$$u(x, t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{1+t} & \text{for } 0 < x \leq 1 + t, \\ 1 & \text{for } x > 1 + t. \end{cases}$$

4. **Example:** Consider the initial value problem

$$(P_2) \begin{cases} u_t + uu_x = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x, 0) = g(x), \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - x & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$$

a. For $\xi \leq 0$, the characteristics have slope 1. Thus

$$u(x, t) \equiv 1 \quad \text{for } x \leq t. \quad (15)$$

b. For $0 < \xi \leq 1$, the characteristics are the lines

$$x = (1 - \xi)t + \xi.$$

Hence,

$$\xi = \frac{x - t}{1 - t},$$

and therefore,

$$u(x, t) = g(\xi(x, t)) = \frac{1 - x}{1 - t} \quad \text{for } t < x \leq 1. \quad (16)$$

c. For $\xi \geq 1$, the characteristics are vertical lines. Consequently,

$$u(x, t) \equiv 0 \quad \text{for } x > 1. \quad (17)$$

- d. The characteristic diagram shows that the characteristics emanating from the interval $[0, 1]$ cross at the point $x = 1, t = 1$, and that no characteristics cross before that time. Since $u \equiv f(\xi)$ along the characteristic through $(\xi, 0)$, the solution must become multivalued, i.e. it “breaks” at $t = 1$. We call $t_b = 1$ the breaking time and $x_b = 1$ the breaking point. The solution

$$u(x, t) = \begin{cases} 1 & \text{for } x \leq t, \\ \frac{1-x}{1-t} & \text{for } t < x \leq 1, \\ 0 & \text{for } x \geq 1, \end{cases}$$

given by (a)-(c) is valid for all x only when $t < 1$.

5. Note that for $t < x < 1$, $u_x(x, t)$ blows up as $t \uparrow 1$. We thus have a “gradient catastrophe” when the solution breaks. In principle, we can compute t_b when it exists. Differentiate (5) with respect to x to get:

$$1 = a'(f(\xi))f'(\xi)\xi_x t + \xi_x = [1 + a'(f(\xi))f'(\xi)t]\xi_x.$$

Therefore,

$$\xi_x = \frac{1}{1 + a'(f(\xi))f'(\xi)t}. \quad (18)$$

And from (9) and (18),

$$u_x = f'(\xi)\xi_x = \frac{f'(\xi)}{1 + a'(f(\xi))f'(\xi)t}. \quad (19)$$

Hence t_b is the smallest time for which

$$1 + a'(f(\xi))f'(\xi)t = 0, \quad (20)$$

for some ξ .

6. **Example:** For (P_2) ,

$$1 + a'(g(\xi))g'(\xi)t = \begin{cases} 1 & \text{for } \xi \leq 0, \\ 1 - t & \text{for } 0 < \xi \leq 1, \\ 1 & \text{for } \xi \geq 1. \end{cases}$$

Hence the breaking time is $t_b = 1$.

7. **Example:** For (P_1) ,

$$1 + a'(f(\xi))f'(\xi)t = \begin{cases} 1 & \text{for } \xi \leq 0, \\ 1 + t & \text{for } 0 < \xi \leq 1, \\ 1 & \text{for } \xi > 1. \end{cases}$$

In this case the solution doesn't break.