Wave Propagation 1

1. By a “wave” we do not necessarily mean something oscillatory, but rather something that propagates in a coherent form with a finite velocity.

2. Earlier in the semester we derived the general conservation law for the density $u$ of some quantity with flux vector $J$:

$$u_t + \text{div} \, J = 0.$$ 

Recall that if the quantity is being convected with velocity field $v$, the flux vector is

$$J = vu.$$  

If the velocity field is a constant vector $c$ then

$$J = J(u) = cu,$$  

and (2) is

$$u_t + c \cdot \nabla u = 0.$$ 

This is called the transport equation. The initial value problem for the transport equation in one space dimension is

$$(P_0) \left\{ \begin{array}{l} u_t + cu_x = 0, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x), \end{array} \right.$$ 

where $c$ is a scalar. You can use the Fourier transform or (as we’ll see below) the method of characteristics to derive the solution

$$u(x,t) = f(x - ct).$$  

You can think of this as representing a signal, moving to the right with velocity $c$.

3. Compare the case of convective transport to that of diffusion with the Fourier flux,

$$J = J(\nabla u) = -k\nabla u,$$  

for some positive constant $k$. This is typical. For diffusion, the flux vector depends on the first derivatives of $u$, and for convective transport, on $u$ itself. You could of course combine the fluxes. The simplest model of convection and diffusion has flux vector

$$J = -k\nabla u + bu.$$  

In this case the conservation law is a convection-diffusion equation

$$u_t - k\Delta u + b \cdot \nabla u = 0.$$
4. Let’s consider the simplest case of nonlinear convective transport in one space dimension:

\[ u_t + J(u)_x = 0. \] \hspace{1cm} (7)

If we set

\[ a(u) = J'(u), \]

then equation (7) becomes

\[ u_t + a(u)u_x = 0. \] \hspace{1cm} (8)

This equation is called quasilinear, since it is linear in its highest-order derivatives.

5. The general, first-order, quasilinear equation in two independent variables is

\[ a(x, t, u)u_x + b(x, t, u)u_t = c(x, t, u). \] \hspace{1cm} (9)

The graph \( z = u(x, t) \) of a solution to (9) is called an integral surface. We’ll solve the initial value problem for (9). This of course will cover the special case of (8).

6. The Method of Characteristics: Consider the initial value problem

\[ (P_1) \left\{ \begin{array}{c} a(x, t, u)u_x + b(x, t, u)u_t = c(x, t, u), \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x, 0) = f(x), \end{array} \right. \]

where \( a, b, c \) are smooth. \((C^1 \text{ is good enough.})\) Let \( u(x, t) \) be the solution. The graph \( S \) of \( z = u(x, t) \) is an integral surface with trace \( z = f(x) \) in the plane \( t = 0 \).

a. In class, we saw that the solution surface \( S \) was “swept out” by curves satisfying the system of autonomous initial value problems,

\[ (i) \begin{cases} \dot{x} = a(x, t, z), \\ x(\xi, 0) = \xi, \end{cases} \quad (ii) \begin{cases} \dot{t} = b(x, t, z), \\ t(\xi, 0) = 0, \end{cases} \quad (iii) \begin{cases} \dot{z} = c(x, t, z), \\ z(\xi, 0) = f(\xi), \end{cases} \]

where the dot indicates differentiation with respect to \( s \). (The set of problems (10) is called the characteristic system.) We solve this system for

\[ \begin{cases} x = x(\xi, s), \\ t = t(\xi, s), \\ z = z(\xi, s). \end{cases} \] \hspace{1cm} (11)

b. We invert the first two of the above equations to obtain \( \xi \) and \( s \) as functions of \( x \) and \( t \):

\[ \begin{cases} \xi = \xi(x, t), \\ s = s(x, t), \end{cases} \] \hspace{1cm} (12)
By (11) and (12), we can think of $z$ as depending on $x$ and $t$. Then,

$$c(x,t,z) = \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial s} = a(x,t,z)\frac{\partial z}{\partial x} + b(x,t,z)\frac{\partial z}{\partial t}.$$ 

Thus

$$u(x,t) = z(\xi(x,t), s(x,t)),$$  \hspace{1cm} (13)

satisfies the PDE (9).

c. By the initial conditions in (10), we see that

$$\xi(x,0) = x \quad \text{and} \quad s(x,0) = 0.$$ 

Therefore,

$$u(x,0) = z(\xi(x,0), s(x,0)) = z(x,0) = f(x).$$

So $u$ satisfies the initial condition.

7. **Definition:** For each fixed $\xi$, the curve traced by $(x(\xi, s), t(\xi, s), z(\xi, s))$ in $\mathbb{R}^3$ is called a characteristic. The curve $(x(\xi, s), t(\xi, s))$ in $\mathbb{R}^2$ is a characteristic projection, though many authors call it a characteristic.

8. **Example:** We’ll use the method of characteristics to solve the initial value problem

$$(P_2) \begin{cases} u_t + uu_x + u = 0 & \text{for } x \in \mathbb{R}, \ t > 0, \\ u(x, 0) = -\frac{x}{2}. \end{cases}$$

The characteristic system is

$$(i) \begin{cases} \dot{x} = z, \\ x(\xi, 0) = \xi, \end{cases} \hspace{1cm} (ii) \begin{cases} \dot{t} = 1, \\ t(\xi, 0) = 0, \end{cases} \hspace{1cm} (iii) \begin{cases} \dot{z} = -z, \\ z(\xi, 0) = -\frac{\xi}{2}. \end{cases}$$  \hspace{1cm} (14)

The solutions to (ii) and (iii) are

$$t = s,$$  \hspace{1cm} (15)

and

$$z = -\frac{\xi}{2} e^{-s}.$$  \hspace{1cm} (16)
respectively. Plug this last expression into (i) to get

\[ x = \frac{\xi}{2}(1 + e^{-s}). \]  

(17)

By (15) and (17),

\[ \xi(x, t) = \frac{2x}{1 + e^{-t}} \quad \text{and} \quad s(x, t) = t. \]  

(18)

Plug these into (16) to get

\[ u(x, t) = z(\xi(x, t), s(x, t)) = -\frac{xe^{-t}}{1 + e^{-t}}. \]  

(19)

9. Example: For the initial value problem \((P_0)\), the characteristic system is

\[
(i) \left\{ \begin{array}{l}
x(t) = c, \\
x(\xi, 0) = \xi,
\end{array} \right. \\
(ii) \left\{ \begin{array}{l}
\dot{t} = 1, \\
t(\xi, 0) = 0,
\end{array} \right. \\
(iii) \left\{ \begin{array}{l}
\dot{z} = 0, \\
z(\xi, 0) = f(\xi).
\end{array} \right.
\]  

(20)

Hence

\[ x = cs + \xi, \quad t = s \]  

(21)

and

\[ z = f(\xi). \]  

(22)

By (21),

\[ \xi = x - ct, \]  

and therefore,

\[ u(x, t) = z(\xi(x, t)) = f(x - ct). \]

10. Note: An extension of the method presented here can be applied to the general first-order, nonlinear equation in any number of independent variables. (See Partial Differential Equations, by L.C. Evans, Partial Differential Equations, by Fritz John, or Partial Differential Equations, by Paul Garabedian. They are all excellent texts, despite the lack of originality in the titles.)