The Wave Equation

1. Consider a wave moving through a continuum (i.e. a continuous body) in $\mathbf{R}^n$. (If $n = 1$, the continuum might be a string, if $n = 2$, a membrane and if $n = 3$, any solid or fluid.) Let $u = u(x, t)$ be a small displacement from equilibrium of the medium in some direction at $x \in \mathbf{R}^n$ and time t.

2. Assume that there is no external force. The disturbance is propagated through the body by adjacent material elements pushing against each other. Let ∂B be a smooth, closed surface bounding a region B. As usual, ν is the outer unit normal vector to ∂B. Let f be the force per unit area exerted by the medium on ∂B at x. As in the case of inviscid fluid flow, we assume that the force acts normally. Thus the net contact force exerted on B through ∂B is

$$-\int_{\partial B} f \cdot \nu \, dS. \tag{1}$$

3. Let $\varrho = \varrho(x)$ be the mass density of the medium. Newton's second law is

$$\frac{d^2}{dt^2} \int_B \varrho u \, dx = -\int_{\partial B} f \cdot \nu \, dS, \tag{2}$$

or

$$\int_B \varrho u_{tt} \, dx + \int_B \operatorname{div} f \, dx = 0. \tag{3}$$

Since (3) holds for arbitrary regions B, it holds pointwise:

$$\varrho u_{tt} + \operatorname{div} f = 0. \tag{4}$$

4. Assume that $\varrho \equiv \varrho_0$, a constant. The simplest, reasonable constitutive equation for f is

$$f = -a \nabla u, \tag{5}$$

for some constant $a > 0$. Thus,

$$\varrho_0 u_{tt} - a \Delta u = 0, \tag{6}$$

or

$$u_{tt} - c^2 \Delta u = 0, \tag{7}$$

where

$$c^2 = \frac{a}{\varrho_0}. \tag{8}$$

The second-order partial differential equation (7) is called the wave equation. The operator

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta, \tag{9}$$

is called the wave operator or the D'Alembertian.