Let $\varphi = \varphi(x, \varepsilon)$ be a smooth function such that

$$\varphi(x_0, 0) = 0.$$

Find a perturbation expansion up to $O(\varepsilon^2)$ for the solution $x$ to the equation

$$\varphi(x, \varepsilon) = 0,$$

where $\varepsilon \ll 1$.

**Method 1**
Start with the Taylor expansion of $\varphi(x, \varepsilon)$ about $(x_0, 0)$:

$$0 = \varphi(x, \varepsilon) = \varphi(x_0, 0) + \varphi_x(x_0, 0)(x - x_0) + \varphi_\varepsilon(x_0, 0)\varepsilon + \frac{1}{2}\varphi_{xx}(x_0, 0)(x - x_0)^2 + \varphi_{x\varepsilon}(x_0, 0)(x - x_0)\varepsilon + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\varepsilon^2 + R(x, \varepsilon), \quad (1)$$

where $R(x, \varepsilon)$ is the remainder. Drop the remainder, set

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3),$$

and

$$\varphi(x_0, 0) = 0,$$

in (1). We get

$$0 = \varphi_x(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2) + \varphi_\varepsilon(x_0, 0)\varepsilon$$

$$+ \frac{1}{2}\varphi_{xx}(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2)^2 + \varphi_{x\varepsilon}(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2)\varepsilon + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\varepsilon^2 + O(\varepsilon^3)$$

$$= [\varphi_x(x_0, 0)x_1 + \varphi_\varepsilon(x_0, 0)]\varepsilon$$

$$+ \left[\varphi_x(x_0, 0)x_2 + \frac{1}{2}\varphi_{xx}(x_0, 0)x_1^2 + \varphi_{x\varepsilon}(x_0, 0)x_1 + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\right]\varepsilon^2$$

$$+ O(\varepsilon^3).$$

Match powers of $\varepsilon$. We get the equations

$$O(\varepsilon) : \quad \varphi_x(x_0, 0)x_1 + \varphi_\varepsilon(x_0, 0) = 0,$$

and

$$O(\varepsilon^2) : \quad \varphi_x(x_0, 0)x_2 + \frac{1}{2}\varphi_{xx}(x_0, 0)x_1^2 + \varphi_{x\varepsilon}(x_0, 0)x_1 + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0) = 0.$$
From these equations we see that the \textit{solvability condition} is
\[
\varphi_x(x_0, 0) \neq 0. \quad (2)
\]
From the $O(\varepsilon)$ equation,
\[
x_1 = -\frac{\varphi_\varepsilon(x_0, 0)}{\varphi_x(x_0, 0)}. \quad (3)
\]
Plug this into the $O(\varepsilon^2)$ equation and solve for
\[
x_2 = -\frac{\varphi_{xx}(x_0, 0)x_1^2 + 2\varphi_{x\varepsilon}(x_0, 0)x_1 + \varphi_{\varepsilon\varepsilon}(x_0, 0)}{2\varphi_x(x_0, 0)}, \quad (4)
\]
where $x_1$ is given by (3).

\textbf{Method 2}

The Implicit Function theorem asserts that if condition (2) is met then the equation
\[
\varphi(x, \varepsilon) = 0
\]
defines a function $x = x(\varepsilon)$ whose graph (in the $x\varepsilon$-plane) passes through the point $(x_0, 0)$.

The perturbation expansion
\[
x = x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + O(\varepsilon^3),
\]
is simply the second-degree Taylor polynomial (with remainder) for $x(\varepsilon)$ about $\varepsilon = 0$. Thus
\[
x_1 = x'(0) \quad \text{and} \quad x_2 = \frac{1}{2}x''(0).
\]
To compute $x_1$, differentiate the equation
\[
\varphi(x(\varepsilon), \varepsilon) = 0
\]
with respect to $\varepsilon$ and solve for $x'$:
\[
x'(\varepsilon) = -\frac{\varphi_\varepsilon(x(\varepsilon), \varepsilon)}{\varphi_x(x(\varepsilon), \varepsilon)}. \quad (5)
\]
Now set $\varepsilon = 0$ to get
\[
x_1 = x'(0) = -\frac{\varphi_\varepsilon(x_0, 0)}{\varphi_x(x_0, 0)}, \quad (6)
\]
which agrees with (3). To complete the solution, differentiate (5) with respect to $\varepsilon$ and set $\varepsilon = 0$. We get
\[
x_2 = \frac{1}{2}x''(0) = -\frac{\varphi_{xx}(x_0, 0)x_1^2 + 2\varphi_{x\varepsilon}(x_0, 0)x_1 + \varphi_{\varepsilon\varepsilon}(x_0, 0)}{2\varphi_x(x_0, 0)}, \quad (7)
\]
where $x_1$ is given by (6). This agrees with (4). Note that in both approaches to the problem, the solvability condition on the derivatives is
\[
\varphi_x(x_0, 0) \neq 0.
\]