Exam 1 Solutions

1. See the first example in the notes on singular perturbation.

2a. Let Θ be temperature and $\mathcal{T}$ time. By comparing terms we see that

$$[q] = \Theta\mathcal{T}^{-1}, \quad [\theta] = \Theta \quad \text{and} \quad [k] = \mathcal{T}^{-1}.$$

2b. Two time scales are $t_1 = \theta/q$ and $t_2 = 1/k$.

2c. When the heat loss term dominates, the appropriate time scale is $t_2 = 1/k$. We therefore set

$$\tau = kt \quad \text{and} \quad x(\tau) = \frac{X(t)}{X_f}.$$

In terms of τ and x the initial value problem is

$$\begin{cases} \dot{x} = ae^{-b/x} - (x-1), \\ x(0) = x_0, \end{cases}$$

where the dot indicates differentiation with respect to τ and $a = q/kX_f$, $b = \theta/X_f$ and $x_0 = X_0/X_f$ are dimensionless.

3a. Assume an expansion

$$m = 1 + \varepsilon m_1 + O(\varepsilon^2),$$

for the root of p near $x = 1$. Plug this into the equation

$$p(m) \equiv m^2 + (3+\varepsilon)m + 2 = 0,$$

and match powers of ε. At $O(1)$ you'll just get $0 = 0$. At $O(\varepsilon)$ the equation is

$$2m_1 - 3m_1 - 1 = 0,$$

so that $m_1 = -1$. Thus, to $O(\varepsilon)$, the root is

$$m = 1 - \varepsilon.$$

3b. With $\varepsilon = 0$, we solve $p(x) = 0$ for $x_0 = 1$. (Remember that we are approximating the root near 1.) We set

$$F(x) = x^2 - 3x + 2,$$

and

$$G(x) = \varepsilon x.$$

The equation for x_1 is

$$F(x_1) = G(1),$$

or

$$x_1^2 - 3x_1^2 + 2 - \varepsilon = 0.$$

By the quadratic formula, the root near 1 is

$$\begin{aligned} x_1 &= \frac{3 - \sqrt{9 - 4(2-\varepsilon)}}{2} \\ &= \frac{3 - \sqrt{1 + 4\varepsilon}}{2} \\ &\approx 1 - \varepsilon. \end{aligned}$$

4a. The dimension matrix A is

$$\begin{array}{c} \\ \mathcal{L} \\ \mathcal{T} \\ \mathcal{M} \\ \Theta \end{array} \begin{array}{c} \begin{array}{ccccc} a & D & \mu & T & k \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}.$$

4b. Since the rank of A is 4, there is one independent dimensionless quantity, π. If

$$\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix},$$

then

$$A\alpha = 0.$$

Thus α lies in the kernel of A, and we may take

$$\pi = \frac{aD\mu}{kT}.$$

By the Buckingham Pi theorem, the physical law $f(a, D, \mu, T, k) = 0$ is equivalent to one of the form

$$F(\pi) = 0.$$

We assume that this implies that π is some constant C. Hence,

$$D = C\,\frac{kT}{a\mu}.$$