1. Suppose that there is a unit-free law
\[ g(P, l, m, t, \rho) = 0, \]
where \( P \) is pressure, \( l \) length, \( m \) mass, \( t \) time and \( \rho \) density. Show there is an equivalent law of the form
\[ G\left(\frac{l^3 \rho}{m}, \frac{t^6 P^3}{m^2 \rho}\right) = 0. \]

2. Consider the two-point boundary value problem
\[
(P_1) \begin{cases}
\epsilon y'' + 2y' + y = 0 & \text{for } 0 < x < 1, \\
y(0) = 0, \\
y(1) = 1,
\end{cases}
\]
where \( 0 < \epsilon \ll 1 \).

a. Find an approximate solution valid uniformly on \([0, 1]\) as \( \epsilon \to 0 \).

b. In what sense is the approximation valid “uniformly” on \([0, 1]\)?

3. Let
\[ I(\lambda) = \int_0^1 e^{\lambda(2t-t^2)} \sqrt{1+t} \, dt. \]
Show that to leading order,
\[ I(\lambda) \sim e^{\lambda} \left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} \quad \text{as} \quad \lambda \to \infty. \]

4. Does the problem
\[
\begin{cases}
u'' + u' - 2u = f(x) & \text{for } 0 < x < 1, \\
u(0) = 0, \\
u'(1) = 0.
\end{cases}
\]
have a Green’s function? If you think so, find it. If you think not, explain why.
5. A substance is confined in a thin tube between $x = 0$ and $x = L$. The density of the substance at the point $x$ at time $t$ is $u(x, t)$. It moves by diffusion only (with diffusion coefficient $D > 0$), and is depleted by a chemical reaction at a rate proportional to its density, (with constant of proportionality $c > 0$).

a. Suppose that the ends of the tube are sealed, so that the substance cannot diffuse across the endpoints $x = 0$ and $x = L$, and that the initial density is $u(x, 0) = f(x)$. Formulate an initial-boundary value problem for $u$.

b. Solve the problem from part (a) with $L = 1$, $D = 1$ and $c = 2$.

6. Let

$$G(x, t) = (4\pi Dt)^{-\frac{1}{2}} e^{-\frac{x^2}{4Dt}},$$

where $D > 0$. Derive the representation

$$u(x, t) = \int_{-\infty}^{\infty} G(x - y, t)g(y) dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s)f(y, s) dy ds,$$

for the solution to the linear diffusion problem

$$\begin{cases} 
  u_t = Du_{xx} + f(x, t), & \text{for } -\infty < x < \infty, t > 0, \\
  u(x, 0) = g(x). 
\end{cases}$$

Assume what you want about $f$ and $g$.

7. Consider the initial value problem

$$\begin{cases} 
  (P_3) \quad u_t + u^3u_x = 0 & \text{for } -\infty < x < \infty, t > 0, \\
  u(x, 0) = \varphi(x), 
\end{cases}$$

where

$$\varphi(x) = \begin{cases} 
  1 & \text{for } x < 0, \\
  1 - x^2 & \text{for } 0 \leq x \leq 1, \\
  0 & \text{for } x > 1. 
\end{cases}$$

a. Determine the breaking time $t_b$.

b. Use the Rankine-Hugoniot condition to determine a solution to $(P_3)$ for $t > t_b$. 