Do three of the first four and three of the second four problems.

1. Consider a very long rod \((x > 0)\) that is laterally insulated and where heat is flowing in the \(x\)-direction; the temperature \(u = u(x, t)\) is governed by the one-dimensional diffusion (heat) equation. The diffusivity is \(k = 0.007\) \(\text{cm}^2\text{sec}^{-1}\). The initial temperature of the rod is 7000 degrees F, and the boundary \(x = 0\) is maintained at zero degrees for all time \(t > 0\). If at some fixed time \(\tau\) we measure the gradient at \(x = 0\) and find \(u_x(0, \tau) = 3.7(10)^{-4}\) deg cm\(^{-1}\), what is \(\tau\)?

2. Consider the following reaction-convection-diffusion equation for \(u(x, t)\):

\[
\frac{\partial}{\partial t} \left((1 + b)u - au^2\right) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}, \tag{1}
\]

where \(0 < a < b\). Show that there exists a traveling wave solution to (1) of the form \(u = U(z)\), \(z = x - ct\), for some wave speed \(c > 0\), where the wave shape \(U = U(z)\) satisfies the boundary conditions

\[U(-\infty) = 1, \quad U(+\infty) = 0.\]

3. A fluid of constant density \(\rho_0\) is flowing through a tube of length \(L\) that has a variable cross-sectional area \(A(x)\), \(0 \leq x \leq L\). The variation in \(A(x)\) is small so that the flow can be considered one dimensional. Let \(u = u(x, t)\) denote the (Eulerian) velocity of the fluid. Derive, from first principles, a mass conservation law for the fluid motion and simplify it as much as possible.

4. Consider the system of equations for \(u = u(x, t)\) and \(v = v(x, t)\) on the domain \(D\): \(0 < x < 1, \ t > 0\):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{\partial u}{\partial x} - au, \\
\frac{\partial v}{\partial t} &= -\frac{\partial v}{\partial x} + u,
\end{align*}
\]

where \(a > 0\) is a constant. Initial and boundary conditions are given by

\[
\begin{align*}
u(x, 0) &= v(x, 0) = 0, \quad 0 \leq x \leq 1, \\
u(0, t) &= b, \quad t > 0,
\end{align*}
\]

where \(b > 0\) is a constant. Find analytic formulas for the solution in the domain \(D\).
5. Let $P$ be the power required to keep a ship of length $L$ moving at a constant speed $V$. Suppose that $P$ depends on the density $\rho$ of the water, the acceleration of gravity $g$, the viscosity of the water $\nu$ (with dimension length²/time) as well as on $V$ and $L$. Show that for some function $f$,

$$P = \rho L^2 V^3 f(Fr, Re),$$

where Fr and Re are the Froude and Reynolds numbers:

$$Fr = \frac{V}{\sqrt{gL}}, \quad Re = \frac{VL}{\nu}.$$

6. The complementary error function is

$$\text{erfc} (\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-t^2} dt.$$

Show that to leading order,

$$\text{erfc} (\lambda) \sim \frac{1}{\sqrt{\pi}} \frac{e^{-\lambda^2}}{\lambda} \quad \text{as} \quad \lambda \to \infty.$$

7. Consider the two-point boundary value problem

$$(P) \begin{cases} y'' + 4y = f(x) & \text{for } 0 < x < \pi, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases}$$

where $f$ is continuous on $[0, \pi]$. Recall that the Green’s function $G$, if it exists, gives an integral representation of the solution $y$:

$$y(x) = \int_0^\pi G(x, \xi) f(\xi) d\xi.$$

Is there a Green’s function for (P)? If there is, find it. If there isn’t, explain how you know.

8. A mass $m$ moves in the $xy$-plane subject to a central force field with potential

$$V = -\frac{k}{\sqrt{x^2 + y^2}},$$

where $k > 0$ is a constant. Show that the Lagrangian in polar coordinates is

$$L(r, \theta) = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}.$$

Use Hamilton’s principle to derive the equations of motion.