Section 2.2, Problem 10

The relativistic orbit equation is

$$\frac{d^2u}{d\varphi^2} + u = a(1 + \varepsilon u^2), \tag{1}$$

where $\varepsilon \ll 1$. With $\varepsilon = 0$ we get the Newtonian description,

$$\frac{d^2u}{d\varphi^2} + u = a. \tag{2}$$

The solution to (2) is

$$u = u_0 = a[1 + e\cos(\varphi - \varphi_0)], \tag{3}$$

where e is the eccentricity and φ_0 the position of the perihelion. Without loss of generality, we may set

$$\varphi_0 = 0. \tag{4}$$

We expand the solution to (1) as a trigonometric polynomial:

$$u = a_0 + a_1\cos(\rho\varphi) + a_2\cos(2\rho\varphi) + \cdots, \tag{5}$$

and assume that

$$\rho = 1 + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + \cdots, \tag{6}$$

and

$$a_k = a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)} + \cdots, \tag{7}$$

for $k \geq 0$. When $\varepsilon = 0$, the expressions for u given in (5) and (3) must coincide. Hence,

$$a_0^{(0)} = a, \quad a_1^{(0)} = ae, \tag{8}$$

and

$$a_k^{(0)} = 0 \text{ for } k \geq 2. \tag{9}$$

Plug the expansion (5) for u into equation (1). You get

$$a_0 + a_1(1-\rho^2)\cos(\rho\varphi) + \cdots \;=\; a + a\varepsilon\left[a_0 + a_1\cos(\rho\varphi) + \cdots\right]^2. \tag{10}$$

Equate the first degree trigonometric terms:

$$a_1(1-\rho^2)\cos(\rho\varphi) = 2a\varepsilon a_0 a_1\cos(\rho\varphi), \tag{11}$$

and hence,

$$1 - \rho^2 = 2a\varepsilon a_0. \tag{12}$$

Plug (6) into the left-hand side of (12), and (7) into the right-hand side:

$$1 - \left(1 + \varepsilon\rho^{(1)} + \cdots\right)^2 = 2a\varepsilon \left[a_0^{(0)} + \varepsilon a_0^{(1)} + \cdots\right] \tag{13}$$

Thus,

$$\rho^{(1)} = -a a_0^{(0)} = -a^2. \tag{14}$$

Put (14) into (6) and drop the $O(\varepsilon^2)$ terms to obtain

$$\rho = 1 - \varepsilon a^2. \tag{15}$$

By (8) and (9), a_0 and a_1 are $O(1)$ and $a_k = O(\varepsilon)$ for $k \geq 2$ as $\varepsilon \downarrow 0$. Thus, to $O(1)$,

$$u = a_0 + a_1 \cos(\rho\varphi) = a_0 + a_1 \cos\left[(1 - \varepsilon a^2)\varphi\right]. \tag{16}$$

The frequency of the approximate solution (16) is $1 - \varepsilon a^2$. Hence the period is

$$\frac{2\pi}{1 - \varepsilon a^2} \approx 2\pi(1 + \varepsilon a^2). \tag{17}$$

The first perihelion occurred at $\varphi_0 = 0$. According to the approximation (17), the next one will occur at $\varphi = 2\pi(1 + \varepsilon a^2)$. Thus, to $O(\varepsilon)$, the *angular* separation is $2\pi\varepsilon a^2$.