Normed Linear Spaces over $\mathbf{C}$ and $\mathbf{R}$

1. The field $\mathbf{F}$ of scalars will always be $\mathbf{C}$ or $\mathbf{R}$.

2. **Definition:** A linear space over the field $\mathbf{F}$ of scalars is a set $V$ satisfying

   a. $V$ is closed under vector addition: For $u$ and $v$ in $V$, $u + v$ is in $V$ also.

   b. Vector addition is commutative and associative: For all $u$, $v$ and $w$ in $V$,
      
      $$u + v = v + u,$$
      
      $$u + (v + w) = (u + v) + w.$$

   c. There is a zero element (denoted 0) in $V$, such that $v + 0 = v$ for all $v$ in $V$.

   d. For each $v$ in $V$, there an additive inverse $-v$ such that $v + (-v) = 0$. (Note: We usually write $u - v$ instead of $u + (-v)$.)

   e. $V$ is closed under scalar multiplication: For $\alpha \in \mathbf{F}$ and $u \in V$, $\alpha u \in V$.

   f. Scalar multiplication is associative and distributive: For all $\alpha$ and $\beta$ in $\mathbf{F}$ and $u$ and $w$ in $V$,
      
      $$\alpha(\beta u) = (\alpha\beta)u,$$
      
      $$(\alpha + \beta)u = \alpha u + \beta u,$$
      
      $$\alpha(u + w) = \alpha u + \alpha w.$$  

   g. $1v = v$ for all $v$ in $V$.

3. **Example:** $\mathbf{R}^n$, with the usual operations, is a vector space over $\mathbf{R}$.

4. **Example:** $\mathbf{C}^n$, with the usual operations, is a vector space over $\mathbf{C}$.

5. **Note:** Instead of the previous two examples, we could have simply stated that $\mathbf{F}^n$, with the usual operations, is a vector space over $\mathbf{F}$.

6. **Example:** The set $C[a, b]$ of $\mathbf{F}$-valued continuous functions defined on $[a, b]$ is a linear space over $\mathbf{F}$. (Note: Elements of $C[a, b]$ are continuous from the right at $a$ and from the left at $b$.)

7. **Example:** The set $C^k[a, b]$ of $\mathbf{F}$-valued $k$-times continuously differentiable functions defined on $[a, b]$, is a linear space over $\mathbf{F}$. (Again, the derivatives are taken from the right at $a$ and from the left at $b$.)
8. **Example:** Let $B \subseteq \mathbb{R}^n$. The set $L^1(B)$ of functions $f : \mathbb{R}^n \to \mathbb{F}$ satisfying
\[
\int_B |f(x)| \, dx < \infty,
\] (1)
is a linear space over $\mathbb{F}$.

9. **Example:** Let $B \subseteq \mathbb{R}^n$. The set $L^2(B)$ of functions $f : \mathbb{R}^n \to \mathbb{F}$ satisfying
\[
\int_B |f(x)|^2 \, dx < \infty,
\] (2)
is a linear space over $\mathbb{F}$.

10. **Definition:** Let $V$ be a linear space. If $U \subseteq V$ is closed under vector addition and scalar multiplication, then $U$ is a *subspace* of $V$. A subspace is itself a linear space.

11. **Example:** Let $V = \mathbb{R}^3$. If $U$ is a subspace of $V$, then either
a. $U = \mathbb{R}^3$,
b. $U$ is a plane through the origin,
c. $U$ is a line through the origin,
d. $U = \{0\}$.

12. **Example:** Let $[a, b]$ be a finite interval. The set $C[a, b]$ is a subspace of $L^1[a, b]$.

13. **Definition:** A norm $\| \|$ on a linear space $V$ is a mapping from $V$ to $\mathbb{R}$ satisfying
a. $\|v\| \geq 0$ for all $v \in V$.
b. $\|v\| = 0$ if and only if $v = 0$.
c. $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$ and $v \in V$.
d. The triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u$ and $v$ in $V$.

The norm assigns to a vector a length or magnitude.

14. The distance between vectors $v$ and $w$ in a normed linear space $V$ is $\|v - w\|$. The (closed) ball about $v$ of radius $r$ is
\[
B(v, r) = \{ w \in V \mid \|v - w\| \leq r \}.
\]
If you replace “less than or equal to” with “less than,” you get the *open* ball.

15. **Example:** $\mathbb{F}^n$ is a normed linear space with
\[
\|z\| = |z| = \left\{ \sum |z_i|^2 \right\}^{\frac{1}{2}}.
\] (3)
16. **Note**: There can be more than one norm on a linear space. For example

\[ \|z\| = |z_1| + \cdots + |z_n|, \]  

(4)

and

\[ \|z\| = \max_{1 \leq i \leq n} |z_i|, \]  

(5)

are also norms on \( \mathbb{F}^n \).

17. **Example**: \( C[a, b] \) is a normed linear space with the maximum (or \( L^\infty \)) norm

\[ \|f\| = \max_{[a, b]} |f(x)|. \]  

(6)

18. **Example**: \( C^k[a, b] \) is a normed linear space with

\[ \|f\| = \sum_{j=1}^{k} \max_{[a, b]} |f^{(j)}(x)|. \]  

(7)

19. **Example**: \( L^1(B) \) is a normed linear space with

\[ \|f\| = \int_B |f(x)| \, dx. \]  

(8)

20. **Example**: \( L^2(B) \) is a normed linear space with

\[ \|f\| = \left\{ \int_B |f(x)|^2 \, dx \right\}^{\frac{1}{2}}. \]  

(9)

21. **Definition**: A sequence \( \{v_k\} \) of vectors in a normed linear space \( V \) is convergent if there is a \( v \in V \) such that

\[ \|v_k - v\| \to 0 \quad \text{as} \quad k \to \infty. \]  

(10)

We say that \( \{v_k\} \) converges to \( v \) and write

\[ \lim_{k \to \infty} v_k = v, \]

or

\[ v_k \to v \quad \text{as} \quad k \to \infty. \]
22. **Definition:** A sequence \( \{v_k\} \) of vectors in a normed linear space \( V \) is Cauchy convergent if
\[
\|v_m - v_n\| \to 0 \quad \text{as} \quad m, n \to \infty.
\] (11)

23. **Definition:** A normed linear space is complete if all Cauchy convergent sequences are convergent. A complete normed linear space is called a Banach space.

24. \( C[a, b] \), \( C^k[a, b] \), \( L^1(B) \) and \( L^2(B) \) are all Banach spaces with respect to the given norms.

25. **Example:** Let \( V \) be the set \( C[0, 2] \) of real-valued functions with norm
\[
\|f\| = \int_0^2 |f(x)| \, dx.
\] (12)

Although \( V \) is a normed linear space, it is not a Banach space. To see this, let
\[
\phi_k(x) = \begin{cases} x^k & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } 1 \leq x \leq 2, \end{cases}
\]
for integers \( k \geq 1 \). Clearly, \( f_k \in V \). Since
\[
\|f_m - f_n\| = \left| \frac{1}{n} - \frac{1}{m} \right| \to 0 \quad \text{as} \quad m, n \to \infty,
\]
the sequence \( \{f_k\} \) is *Cauchy convergent* in \( V \). Suppose that there were a function \( f \) in \( V \) such that
\[
\|f_m - f\| \to 0 \quad \text{as} \quad m \to \infty.
\]
It would have to be that
\[
f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } 1 \leq x \leq 2, \end{cases}
\]
which is discontinuous, and hence not in \( V \). Thus the Cauchy convergent sequence \( \{f_k\} \) is not convergent (in the norm on \( V \)), and \( V \) is therefore not a Banach space.

26. Why should you bother with the distinction between Banach spaces and incomplete normed linear spaces? Many equations are solved by iterative procedures: We generate a sequence \( \{v_k\} \) of approximate solutions, hoping it will converge to a solution \( v \). How do you prove convergence? You don’t know if \( v \) even exists. If the \( v_k \) live in a Banach space \( V \) with norm \( \| \| \), it is only necessary to show that the sequence is Cauchy convergent. Then (by the definition of completeness) you are guaranteed the existence of a \( v \in V \) such that \( v_k \to v \) as \( k \to \infty \).

27. A norm assigns a magnitude to a vector. We’d like a notion of angle as well. To this end, we introduce inner products—generalizations of the dot product on \( \mathbb{R}^3 \).
28. **Definition:** An inner product on a linear space $V$ over $F$ is a mapping $\langle , \rangle$ from $V \times V$ to $F$ satisfying

a. $\langle v, v \rangle \geq 0$ for all $v \in V$.

b. $\langle v, v \rangle = 0$ if and only if $v = 0$.

c. $\langle u, v \rangle = \langle v, u \rangle^*$ for all $u$ and $v$ in $V$.

d. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ for all $\alpha \in F$ and $u$ and $v$ in $V$.

e. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u$, $v$ and $w$ in $V$.

29. **Note:** If $V$ is a linear space over $R$, then $\langle u, v \rangle$ is a real number. In this case (c) becomes

$$\langle u, v \rangle = \langle v, u \rangle,$$

for all $u$ and $v$ in $V$.

30. **Example:** $F^n$ is an inner product space: For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, in $F^n$,

$$\langle x, y \rangle = x_1 y_1^* + \cdots + x_n y_n^*.$$  \hspace{1cm} (13)

Note that when $F = R$, this reduces to the usual dot product on $R^n$:

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$  \hspace{1cm} (14)

31. **Example:** For a vector of positive weights $w = (w_1, \ldots, w_n)$,

$$\langle x, y \rangle = w_1 x_1 y_1^* + \cdots + w_n x_n y_n^*,$$  \hspace{1cm} (15)

is an inner product on $F^n$.

32. **Example:** $L^2(B)$ is an inner product space with

$$\langle f, g \rangle = \int_B f(x) g(x)^* \, dx.$$  \hspace{1cm} (16)

33. **Example:** Let $w : R^n \rightarrow R$ be bounded, real-valued and positive on $B$. Then for $f$ and $g$ taking $R^n$ to $R$,

$$\langle f, g \rangle = \int_B w(x) f(x) g(x)^* \, dx,$$  \hspace{1cm} (17)

defines an inner product.

34. Let $V$ be an inner product space. For $v \in V$, set

$$\|v\| = \sqrt{\langle v, v \rangle}.$$  \hspace{1cm} (18)
The notation suggests that (18) defines a norm on $V$. We’ll show that this is the case.

35. **The Cauchy-Schwarz Inequality:** For all $u$ and $v$ in $V$,

$$|\langle u, v \rangle| \leq \|u\| \|v\|. \quad (19)$$

36. It follows easily from (19) that

$$\|u + v\| \leq \|u\| + \|v\|. \quad (20)$$

From (20) and properties (a), (b) and (d) of the inner product, we see that (18) really does define a norm. Thus an inner product space is automatically a normed linear space.

37. If the inner product space is $L^2(B)$, then the Cauchy-Schwarz inequality becomes

$$\left| \int_B f(x)g(x) \, dx \right| \leq \left\{ \int_B |f(x)|^2 \, dx \right\}^{\frac{1}{2}} \left\{ \int_B |g(x)|^2 \, dx \right\}^{\frac{1}{2}}.$$

38. An inner product space has a richer geometry than a space that is merely normed. In a normed space we only have length. In an inner product space we have length and angle: We define the angle $\theta$ between $u$ and $v$ in an inner product space by

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\|\|v\|}. \quad (21)$$

This generalizes the formula for the angle between two vectors in $\mathbb{C}^3$.

39. Vectors $u$ and $v$ in an inner product space are called orthogonal if

$$\langle u, v \rangle = 0.$$

40. **Definition:** An inner product space that is complete with respect to the norm (18) is called a Hilbert space.

41. $\mathbb{C}^n$ and $L^2(B)$ are Hilbert spaces with the given inner products. In a sense, there are no more (separable) Hilbert spaces. Any $n$-dimensional Hilbert space is an algebraic and geometric copy of $\mathbb{C}^n$, and any infinite-dimensional (separable) Hilbert space is an algebraic and geometric copy of $L^2(B)$. 