Normed Linear Spaces over **C** and **R**

1. The field **F** of scalars will always be **C** or **R**.

2. **Definition**: A linear space over the field **F** of scalars is a set V satisfying

a. V is closed under vector addition: For u and v in V, $u+v$ is in V also.

b. Vector addition is commutative and associative: For all u, v and w in V,

$$u+v=v+u,$$

$$(u+v)+w=u+(v+w).$$

c. There is a zero element (denoted 0) in V, such that $v+0=v$ for all v in V.

d. For each v in V, there an additive inverse $-v$ such that $v+(-v)=0$. (Note: We usually write $u-v$ instead of $u+(-v)$.)

e. V is closed under scalar multiplication: For $\alpha \in \mathbf{F}$ and $u \in V$, $\alpha u \in V$.

f. Scalar multiplication is associative and distributive: For all α and β in **F** and u and w in V,

$$\alpha(\beta u)=(\alpha\beta)u,$$

$$(\alpha+\beta)u=\alpha u+\beta u,$$

$$\alpha(u+w)=\alpha u+\alpha w.$$

g. $1\,v=v$ for all v in V.

3. **Example**: $\mathbf{R}^n$, with the usual operations, is a vector space over **R**.

4. **Example**: $\mathbf{C}^n$, with the usual operations, is a vector space over **C**.

5. **Note**: Instead of the previous two examples, we could have simply stated that $\mathbf{F}^n$, with the usual operations, is a vector space over **F**.

6. **Example**: The set $C[a,b]$ of **F**-valued continuous functions defined on $[a,b]$ is a linear space over **F**. (Note: Elements of $C[a,b]$ are continuous from the right at a and from the left at b.)

7. **Example**: The set $C^k[a,b]$ of **F**-valued k-times continuously differentiable functions defined on $[a,b]$, is a linear space over **F**. (Again, the derivatives are taken from the right at a and from the left at b.)

8. Example: Let $B \subseteq \mathbf{R}^n$. The set $L^1(B)$ of functions $f : \mathbf{R}^n \to \mathbf{F}$ satisfying

$$\int_B |f(x)|\, dx < \infty, \tag{1}$$

is a linear space over $\mathbf{F}$.

9. Example: Let $B \subseteq \mathbf{R}^n$. The set $L^2(B)$ of functions $f : \mathbf{R}^n \to \mathbf{F}$ satisfying

$$\int_B |f(x)|^2\, dx < \infty, \tag{2}$$

is a linear space over $\mathbf{F}$.

10. Definition: Let V be a linear space. If $U \subseteq V$ is closed under vector addition and scalar multiplication, then U is a *subspace* of V. A subspace is itself a linear space.

11. Example: Let $V = \mathbf{R}^3$. If U is a subspace of V, then either

a. $U = \mathbf{R}^3$,

b. U is a plane through the the origin,

c. U is a line through the origin,

d. $U = \{0\}$.

12. Example: Let $[a, b]$ be a finite interval. The set $C[a, b]$ is a subspace of $L^1[a, b]$.

13. Definition: A norm $\| \;\|$ on a linear space V is a mapping from V to $\mathbf{R}$ satisfying

a. $\|v\| \geq 0$ for all $v \in V$.

b. $\|v\| = 0$ if and only if $v = 0$.

c. $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbf{C}$ and $v \in V$.

d. The triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all u and v in V.

The norm assigns to a vector a length or magnitude.

14. The distance between vectors v and w in a normed linear space V is $\|v - w\|$. The (closed) ball about v of radius r is

$$B(v, r) = \{w \in V \;\mid\; \|v - w\| \leq r\}.$$

If you replace "less than or equal to" with "less than," you get the *open* ball.

15. Example: $\mathbf{F}^n$ is a normed linear space with

$$\|z\| = |z| = \left\{\, |z_1|^2 + \cdots |z_n|^2 \,\right\}^{\frac{1}{2}}. \tag{3}$$

16. **Note**: There can be more than one norm on a linear space. For example

$$\|z\| = |z_1| + \cdots + |z_n|, \tag{4}$$

and

$$\|z\| = \max_{1 \le i \le n} |z_i|, \tag{5}$$

are also norms on $\mathbf{F}^n$.

17. **Example**: $C[a, b]$ is a normed linear space with the maximum (or L^∞) norm

$$\|f\| = \max_{[a,b]} |f(x)|. \tag{6}$$

18. **Example**: $C^k[a, b]$ is a normed linear space with

$$\|f\| = \sum_{j=1}^{k} \max_{[a,b]} |f^{(j)}(x)|. \tag{7}$$

19. **Example**: $L^1(B)$ is a normed linear space with

$$\|f\| = \int_B |f(x)|\, dx. \tag{8}$$

20. **Example**: $L^2(B)$ is a normed linear space with

$$\|f\| = \left\{ \int_B |f(x)|^2\, dx \right\}^{\frac{1}{2}}. \tag{9}$$

21. **Definition**: A sequence $\{v_k\}$ of vectors in a normed linear space V is convergent if there is a $v \in V$ such that

$$\|v_k - v\| \to 0 \quad \text{as} \quad k \to \infty. \tag{10}$$

We say that $\{v_k\}$ converges to v and write

$$\lim_{k \to \infty} v_k = v,$$

or

$$v_k \to v \quad \text{as} \quad k \to \infty.$$

22. **Definition**: A sequence $\{v_k\}$ of vectors in a normed linear space V is Cauchy convergent if

$$\|v_m - v_n\| \to 0 \quad \text{as} \quad m, n \to \infty. \tag{11}$$

23. **Definition**: A normed linear space is complete if all Cauchy convergent sequences are convergent. A complete normed linear space is called a Banach space.

24. $C[a, b]$, $C^k[a, b]$, $L^1(B)$ and $L^2(B)$ are all Banach spaces with respect to the given norms.

25. **Example**: Let V be the set $C[0, 2]$ of real-valued functions with norm

$$\|f\| = \int_0^2 |f(x)|\, dx. \tag{12}$$

Although V is a normed linear space, it is not a Banach space. To see this, let

$$\phi_k(x) = \begin{cases} x^k & \text{for } 0 \le x \le 1, \\ 1 & \text{for } 1 \le x \le 2, \end{cases}$$

for integers $k \ge 1$. Clearly, $f_k \in V$. Since

$$\|f_m - f_n\| = \left| \frac{1}{n} - \frac{1}{m} \right| \to 0 \quad \text{as} \quad m, n \to \infty,$$

the sequence $\{f_k\}$ is *Cauchy convergent* in V. Suppose that there were a function f in V such that

$$\|f_m - f\| \to 0 \quad \text{as} \quad m \to \infty.$$

It would have to be that

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } 1 \le x \le 2, \end{cases}$$

which is discontinuous, and hence not in V. Thus the Cauchy convergent sequence $\{f_k\}$ is not convergent (in the norm on V), and V is therefore not a Banach space.

26. Why should you bother with the distinction between Banach spaces and incomplete normed linear spaces? Many equations are solved by iterative procedures: We generate a sequence $\{v_k\}$ of approximate solutions, hoping it will converge to a solution v. How do you prove convergence? You don't know if v even exists. If the v_k live in a Banach space V with norm $\|\ \|$, it is only necessary to show that the sequence is Cauchy convergent. Then (by the definition of completeness) you are guaranteed the existence of a $v \in V$ such that $v_k \to v$ as $k \to \infty$.

27. A norm assigns a magnitude to a vector. We'd like a notion of angle as well. To this end, we introduce inner products—generalizations of the dot product on $\mathbf{R}^3$.

28. **Definition**: An inner product on a linear space V over $\mathbf{F}$ is a mapping $\langle\,,\rangle$ from $V \times V$ to $\mathbf{F}$ satisfying

a. $\langle v\,, v\rangle \geq 0$ for all $v \in V$.

b. $\langle v\,, v\rangle = 0$ if and only if $v = 0$.

c. $\langle u\,, v\rangle = \langle v\,, u\rangle^*$ for all u and v in V.

d. $\langle \alpha u\,, v\rangle = \alpha\langle u\,, v\rangle$ for all $\alpha \in \mathbf{F}$ and u and v in V.

e. $\langle u + v\,, w\rangle = \langle u\,, w\rangle + \langle v\,, w\rangle$ for all u, v and w in V.

29. **Note**: If V is a linear space over $\mathbf{R}$, then $\langle u\,, v\rangle$ is a real number. In this case (c) becomes

$$\langle u\,, v\rangle = \langle v\,, u\rangle, \quad \text{for all } u \text{ and } v \text{ in } V.$$

30. **Example**: $\mathbf{F}^n$ is an inner product space: For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, in $\mathbf{F}^n$,

$$\langle x\,, y\rangle = x_1 y_1^* + \cdots + x_n y_n^*. \tag{13}$$

Note that when $\mathbf{F} = \mathbf{R}$, this reduces to the usual dot product on $\mathbf{R}^n$:

$$\langle x\,, y\rangle = x \cdot y = x_1 y_1 + \cdots + x_n y_n. \tag{14}$$

31. **Example**: For a vector of positive weights $w = (w_1, \ldots, w_n)$,

$$\langle x\,, y\rangle = w_1 x_1 y_1^* + \cdots + w_n x_n y_n^*, \tag{15}$$

is an inner product on $\mathbf{F}^n$.

32. **Example**: $L^2(B)$ is an inner product space with

$$\langle f\,, g\rangle = \int_B f(x) g(x)^*\, dx. \tag{16}$$

33. **Example**: Let $w : \mathbf{R}^n \to \mathbf{R}$ be bounded, real-valued and positive on B. Then for f and g taking $\mathbf{R}^n$ to $\mathbf{R}$,

$$\langle f\,, g\rangle = \int_B w(x) f(x) g(x)^*\, dx, \tag{17}$$

defines an inner product.

34. Let V be an inner product space. For $v \in V$, set

$$\|v\| = \sqrt{\langle v\,, v\rangle}. \tag{18}$$

The notation suggests that (18) defines a norm on V. We'll show that this is the case.

35. **The Cauchy-Schwarz Inequality:** For all u and v in V,

$$|\langle u\,,v\rangle| \leq \|u\|\,\|v\|. \tag{19}$$

36. It follows easily from (19) that

$$\|u+v\| \leq \|u\| + \|v\|. \tag{20}$$

From (20) and properties (a), (b) and (d) of the inner product, we see that (18) really does define a norm. Thus an inner product space is automatically a normed linear space.

37. If the inner product space is $L^2(B)$, then the Cauchy-Schwarz inequality becomes

$$\left|\int_B f(x)g(x)\,dx\right| \leq \left\{\int_B |f(x)|^2\,dx\right\}^{\frac{1}{2}} \left\{\int_B |g(x)|^2\,dx\right\}^{\frac{1}{2}}.$$

38. An inner product space has a richer geometry than a space that is merely normed. In a normed space we only have length. In an inner product space we have length and angle: We define the angle θ between u and v in an inner product space by

$$\cos\theta = \frac{\langle u\,,v\rangle}{\|u\|\|v\|}. \tag{21}$$

This generalizes the formula for the angle between two vectors in $\mathbf{C}^3$.

39. Vectors u and v in an inner product space are called orthogonal if

$$\langle u\,,v\rangle = 0.$$

40. **Definition**: An inner product space that is complete with respect to the norm (18) is called a Hilbert space.

41. $\mathbf{C}^n$ and $L^2(B)$ are Hilbert spaces with the given inner products. In a sense, there are no more (separable) Hilbert spaces. Any n-dimensional Hilbert space is an algebraic and geometric copy of $\mathbf{C}^n$, and any infinite-dimensional (separable) Hilbert space is an algebraic and geometric copy of $L^2(B)$.