Concerning $L^2(I)$

1. $L^2(I)$ is the class of complex-valued functions f defined on the interval I such that

$$\int_I |f(x)|^2 \, dx < \infty. \tag{1}$$

(The integral appearing here and throughout the discussion of $L^2(I)$ is the *Lebesgue* integral, as opposed to the Riemann. For any reasonable function, the two integrals coincide, so we needn't worry about the distinction.)

a. $L^2(I)$ is a linear, or vector space over the field $\mathbf{C}$ of complex scalars. For practical purposes, this means that for any functions f and g in $L^2(I)$, and any complex numbers α and β, the linear combination $\alpha f + \beta g$ lies in $L^2(I)$.

b. The $L^2(I)$ inner product is

$$\langle f \, , \, g \rangle = \int_I f(x)\bar{g}(x) \, dx. \tag{2}$$

c. The $L^2(I)$ norm is

$$\|f\|_2 = \sqrt{\langle f \, , \, f \rangle} = \left(\int_I |f(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{3}$$

The inner product and norm on $L^2(I)$ were clearly inspired by the standard inner product and norm on $\mathbf{C}^n$.

2. The *Cauchy-Schwarz Inequality* asserts that for f and g in $L^2(I)$,

$$|\langle f \, , \, g \rangle| \le \|f\|_2 \, \|g\|_2. \tag{4}$$

3. A sequence $\{f_k\}$ of functions in $L^2(I)$ is *convergent* if there is an $f \in L^2(I)$ such that

$$\|f_k - f\|_2 \to 0 \quad \text{as} \quad k \to \infty. \tag{5}$$

We say that $\{f_k\}$ converges to f and write

$$\lim_{k \to \infty} f_k = f,$$

or

$$f_k \to f \quad \text{as} \quad k \to \infty.$$

4. A sequence $\{f_k\}$ of functions in $L^2(I)$ is *Cauchy convergent* if

$$\|f_m - f_n\|_2 \to 0 \quad \text{as} \quad m, n \to \infty. \tag{6}$$

5. If a sequence $\{f_k\}$ in $L^2(I)$ is Cauchy convergent, it is also convergent. Thus $L^2(I)$ is *complete* with respect to the norm (3).

6. $L^2(I)$ is a *Hibert space*, that is, a complete inner product space.

7. A set $\{f_k\}$ of vectors in $L^2(I)$ is *orthogonal* if

$$\langle f_i \, , \, f_j \rangle = 0 \quad \text{for } i \neq j.$$

8. The Kronecker delta is

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

9. A set $\{\varphi_k\}$ in $L^2(I)$ is *orthonormal* if

$$\langle \varphi_i \, , \, \varphi_j \rangle = \delta_{ij}.$$

Thus $\{\varphi_k\}$ is orthonormal if it is orthogonal and the φ_k are unit vectors.

10. Note that if $\{f_k\}$ is an orthogonal set of nonzero vectors, then $\{f_k/\|f_k\|_2\}$ is orthonormal.

11. The set $\{\varphi_k\}$ is an orthonormal *basis* of $L^2(I)$ if

a. it is orthonormal and

b. every vector $f \in L^2(I)$ can be represented as a linear combination of the φ_k:

$$f = \sum_k c_k \varphi_k, \tag{7}$$

for scalars c_k.

12. If the sum in (7) is infinite, then equality is understood to hold in the sense of norm convergence:

$$\| f - \sum_{k=1}^{N} c_k \varphi_k \|_2 \to 0 \quad \text{as} \quad N \to \infty.$$

13. For g in $L^2(I)$, and $\mathcal{O} = \{\varphi_k\}$ an orthonormal basis, there are scalars c_k such that

$$g = \sum_k c_k \varphi_k.$$

By taking inner products, we see that $c_m = \langle g\,,\, e_m\rangle$. Thus,

$$g = \sum_k \langle g\,,\, \varphi_k\rangle\, \varphi_k. \tag{8}$$

The sum (8) is called the Fourier series of g with respect to $\mathcal{O}$. The numbers

$$\hat{g}_k = \langle g\,,\, \varphi_k\rangle$$

are the Fourier coefficients.

14. Let $\mathcal{O} = \{\varphi_k\}$ be an orthonormal basis of $L^2(I)$. Then for functions f and g in $L^2(I)$,

$$\langle f\,,\, g\rangle = \sum_k \langle f\,,\, \varphi_k\rangle\, \overline{\langle g\,,\, \varphi_k\rangle}. \tag{9}$$

If $f = g$, this becomes

$$\|g\|_2^2 = \sum_k |\, \langle g\,,\, \varphi_k\rangle\, |^2. \tag{10}$$

Formulae (9) and (10) are versions of Parseval's identity; (10) is a generalization of the Pythagorean theorem.

15. If $\{\varphi_k\}$ is an orthonormal set (not necessarily a basis) in $L^2(I)$, then (10) becomes

$$\|g\|_2^2 \geq \sum_k |\, \langle g\,,\, \varphi_k\rangle\, |^2. \tag{11}$$

This is Bessel's inequality. It can be shown that equality holds in (11) for all $g \in L^2(I)$ if and only if $\{\varphi_k\}$ is a basis of $L^2(I)$.