Concerning $L^2(I)$

1. $L^2(I)$ is the class of complex-valued functions $f$ defined on the interval $I$ such that

$$\int_I |f(x)|^2 \, dx < \infty.$$  \hspace{1cm} (1)

(The integral appearing here and throughout the discussion of $L^2(I)$ is the Lebesgue integral, as opposed to the Riemann. For any reasonable function, the two integrals coincide, so we needn’t worry about the distinction.)

a. $L^2(I)$ is a linear, or vector space over the field $\mathbb{C}$ of complex scalars. For practical purposes, this means that for any functions $f$ and $g$ in $L^2(I)$, and any complex numbers $\alpha$ and $\beta$, the linear combination $\alpha f + \beta g$ lies in $L^2(I)$.

b. The $L^2(I)$ inner product is

$$\langle f, g \rangle = \int_I f(x)\overline{g(x)} \, dx.$$  \hspace{1cm} (2)

c. The $L^2(I)$ norm is

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left( \int_I |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3)

The inner product and norm on $L^2(I)$ were clearly inspired by the standard inner product and norm on $\mathbb{C}^n$.

2. The Cauchy-Schwarz Inequality asserts that for $f$ and $g$ in $L^2(I)$,

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2.$$  \hspace{1cm} (4)

3. A sequence $\{f_k\}$ of functions in $L^2(I)$ is convergent if there is an $f \in L^2(I)$ such that

$$\|f_k - f\|_2 \to 0 \quad \text{as} \quad k \to \infty.$$  \hspace{1cm} (5)

We say that $\{f_k\}$ converges to $f$ and write

$$\lim_{k \to \infty} f_k = f,$$

or

$$f_k \to f \quad \text{as} \quad k \to \infty.$$

4. A sequence $\{f_k\}$ of functions in $L^2(I)$ is Cauchy convergent if

$$\|f_m - f_n\|_2 \to 0 \quad \text{as} \quad m, n \to \infty.$$  \hspace{1cm} (6)
5. If a sequence \( \{f_k\} \) in \( L^2(I) \) is Cauchy convergent, it is also convergent. Thus \( L^2(I) \) is **complete** with respect to the norm (3).

6. \( L^2(I) \) is a **Hilbert space**, that is, a complete inner product space.

7. A set \( \{f_k\} \) of vectors in \( L^2(I) \) is **orthogonal** if

\[
\langle f_i, f_j \rangle = 0 \quad \text{for } i \neq j.
\]

8. The Kronecker delta is

\[
\delta_{ij} = \begin{cases} 
0 & \text{for } i \neq j, \\
1 & \text{for } i = j.
\end{cases}
\]

9. A set \( \{\varphi_k\} \) in \( L^2(I) \) is **orthonormal** if

\[
\langle \varphi_i, \varphi_j \rangle = \delta_{ij}.
\]

Thus \( \{\varphi_k\} \) is orthonormal if it is orthogonal and the \( \varphi_k \) are unit vectors.

10. Note that if \( \{f_k\} \) is an orthogonal set of nonzero vectors, then \( \{f_k/\|f_k\|_2\} \) is orthonormal.

11. The set \( \{\varphi_k\} \) is an **orthonormal basis** of \( L^2(I) \) if

a. it is orthonormal and

b. every vector \( f \in L^2(I) \) can be represented as a linear combination of the \( \varphi_k \):  

\[
f = \sum_k c_k \varphi_k, \tag{7}
\]

for scalars \( c_k \).

12. If the sum in (7) is infinite, then equality is understood to hold in the sense of norm convergence:

\[
\| f - \sum_{k=1}^{N} c_k \varphi_k \|_2 \to 0 \quad \text{as } N \to \infty.
\]

13. For \( g \) in \( L^2(I) \), and \( \mathcal{O} = \{\varphi_k\} \) an orthonormal basis, there are scalars \( c_k \) such that

\[
g = \sum_k c_k \varphi_k.
\]
By taking inner products, we see that $c_m = \langle g, e_m \rangle$. Thus,

$$g = \sum_k \langle g, \varphi_k \rangle \varphi_k. \quad (8)$$

The sum (8) is called the Fourier series of $g$ with respect to $O$. The numbers $\hat{g}_k = \langle g, \varphi_k \rangle$ are the Fourier coefficients.

14. Let $O = \{\varphi_k\}$ be an orthonormal basis of $L^2(I)$. Then for functions $f$ and $g$ in $L^2(I)$,

$$\langle f, g \rangle = \sum_k \langle f, \varphi_k \rangle \overline{\langle g, \varphi_k \rangle}. \quad (9)$$

If $f = g$, this becomes

$$\|g\|_2^2 = \sum_k |\langle g, \varphi_k \rangle|^2. \quad (10)$$

Formulæ (9) and (10) are versions of Parseval’s identity; (10) is a generalization of the Pythagorean theorem.

15. If $\{\varphi_k\}$ is an orthonormal set (not necessarily a basis) in $L^2(I)$, then (10) becomes

$$\|g\|_2^2 \geq \sum_k |\langle g, \varphi_k \rangle|^2. \quad (11)$$

This is Bessel’s inequality. It can be shown that equality holds in (11) for all $g \in L^2(I)$ if and only if $\{\varphi_k\}$ is a basis of $L^2(I)$. 