1. Remember the three main ideas behind Laplace’s method:

   a. For $\lambda \gg 1$, the main contribution to $I(\lambda)$ comes from a small region of the minimizer $t = c$. We can thus replace an integral over $[a, b]$ with an integral over $(c - \varepsilon, c + \varepsilon)$. (Or over $[a, a + \varepsilon]$, etc.)

   b. In the small neighborhood of the minimizer, we can approximate $f(t)$ and $g(t)$ with Taylor polynomials.

   c. We may extend the interval of integration to include any region that only contributes higher-order terms to $I(\lambda)$ as $\lambda \to \infty$.

   We can add one more:

   d. We can use properties of the gamma function to evaluate integrals that arise in asymptotic analysis.

You should be able to use these ideas to derive asymptotic approximations for integrals of the sort

   $$ I(\lambda) = \int_a^b f(t) e^{-\lambda g(t)} \, dt, \quad (1) $$

without recourse to formulae and theorems.

2. Example: Suppose that $g(t)$ attains a strict minimum over $[a, b]$ at $c \in (a, b)$, that $g'(c) = 0$, $g''(c) > 0$, $f(c) = 0$ and $f''(c) \neq 0$. Find the leading order behavior of $I(\lambda)$ as $\lambda \to \infty$. By the usual reasoning,

   $$ I(\lambda) \approx e^{\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{-\lambda [g(t) - g(c)]} \, dt $$

   $$ \approx e^{-\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} \left[ f'(c)(t - c) + \frac{f''(c)}{2}(t - c)^2 \right] e^{-\frac{\lambda}{2} g''(c)(t-c)^2} \, dt $$

   $$ = e^{-\lambda g(c)} \int_{-\infty}^{\infty} \left[ f'(c)s + \frac{f''(c)}{2}s^2 \right] e^{-\frac{\lambda}{2} g''(c)s^2} \, ds $$

   $$ = \frac{f''(c)}{2} e^{-\lambda g(c)} \int_{-\infty}^{\infty} s^2 e^{-\frac{\lambda}{2} g''(c)s^2} \, ds $$

   $$ = f''(c)e^{-\lambda g(c)} \int_0^{\infty} s^2 e^{-\frac{\lambda}{2} g''(c)s^2} \, ds. $$

Make the change of variable

   $$ \frac{\lambda}{2} g''(c)s^2 = u, $$

and use properties of the gamma function to show that

   $$ \int_0^{\infty} s^2 e^{-\frac{\lambda}{2} g''(c)s^2} \, ds = \sqrt{\frac{\pi}{(\lambda g''(c))^3}}. $$
Thus, to leading order,

\[ I(\lambda) \sim f''(c)e^{-\lambda g(c)} \sqrt{\frac{\pi}{(\lambda g''(c))^3}}, \quad \text{as } \lambda \to \infty. \]

3. Example: Suppose that \( g(t) \) attains a strict minimum over \([a, b]\) at \( a \), that \( g'(a) = 0 \), \( g''(a) > 0 \), \( f(a) = 0 \) and \( f'(a) \neq 0 \). Find the leading order behavior of \( I(\lambda) \) as \( \lambda \to \infty \).

\[
I(\lambda) \approx e^{\lambda g(a)} \int_{a}^{a+\epsilon} f(t)e^{-\lambda [g(t)-g(a)]} dt
\approx e^{-\lambda g(a)} \int_{a}^{a+\epsilon} f'(a)(t-a)e^{-\frac{\lambda}{2}g''(a)(t-a)^2} dt
\approx f'(a)e^{-\lambda g(a)} \int_{a}^{\infty} (t-a)e^{-\frac{\lambda}{2}g''(a)(t-a)^2} dt
= f'(a)e^{-\lambda g(a)} \int_{0}^{\infty} se^{-\frac{\lambda}{2}g''(a)s^2} ds
= f'(a)e^{-\lambda g(a)} \frac{1}{\lambda g''(a)}.
\]

Thus, to leading order,

\[ I(\lambda) \sim f'(a) \frac{e^{-\lambda g(a)}}{\lambda g''(a)} \text{ as } \lambda \to \infty. \]

4. You can sometimes use integration by parts to determine asymptotic behavior. Consider, for example, the complementary error function

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt. \]

Then,

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \frac{de^{-t^2}}{-2t} = \frac{2}{\sqrt{\pi}} \left( \frac{e^{-x^2}}{2x} - \int_{x}^{\infty} \frac{e^{-t^2}}{2t^2} dt \right). \quad (2)
\]

Repeat the process:

\[
\int_{x}^{\infty} \frac{e^{-t^2}}{2t^2} dt = \int_{x}^{\infty} \frac{de^{-t^2}}{-4t^3} = \frac{e^{-x^2}}{4x^3} - \int_{x}^{\infty} \frac{3e^{-t^2}}{4t^4} dt. \quad (3)
\]

Plug (2) into (3) to get

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \left( \frac{1}{2x} - \frac{1}{4x^3} \right) + \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \frac{3e^{-t^2}}{4t^4} dt. \quad (4)
\]
You could integrate by parts once more and then do a little calculus to show that
\[
\int_x^\infty \frac{3e^{-t^2}}{4t^4} \, dt = O \left( \frac{e^{-x^2}}{x^4} \right) \quad \text{as } x \to \infty.
\]
Thus,
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \left[ \left( \frac{1}{2x} - \frac{1}{4x^3} \right) + o(x^{-3}) \right],
\]
or
\[
\text{erfc}(x) \sim \frac{2}{\sqrt{\pi}} e^{-x^2} \left( \frac{1}{2x} - \frac{1}{4x^3} \right) \quad \text{as } x \to \infty.
\]
You can do this indefinitely. If you are satisfied with the \( O(x^{-3}) \) expansion, then stop at \( () \).

5. You could also treat the integral from the previous example using Watson’s lemma. Set
\[
s = t - x.
\]
Then
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-t^2} e^{-2xt} \, dt.
\]
Now set
\[
\tau = 2t,
\]
to get
\[
\text{erfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-\frac{\tau^2}{4}} e^{-x\tau} \, d\tau.
\]
Finally, apply Watson’s lemma to the above integral to get
\[
\text{erfc}(x) \sim \frac{2}{\sqrt{\pi}} e^{-x^2} \left( \frac{1}{2x} - \frac{\Gamma(3)}{2!(2x)^3} + \frac{\Gamma(5)}{2!(2x)^5} - \cdots \right) \quad \text{as } x \to \infty.
\]
If we drop the \( O(x^{-5}) \) terms and higher, then \( () \) reduces to \( () \).