Integral Asymptotics 2: Watson's Lemma

1. Consider the integral

$$J(\lambda) = \int_0^{\frac{\pi}{2}} e^{-\lambda \tan^2 \theta}\, d\theta. \tag{1}$$

If we set

$$t = \tan^2 \theta,$$

then the integral becomes

$$J(\lambda) = \int_0^{\infty} \frac{1}{\sqrt{t}} \frac{1}{2(1+t)} e^{-\lambda t}\, dt. \tag{2}$$

This leads us to consider the asymptotic behavior of

$$I(\lambda) = \int_0^{b} t^{\alpha} h(t) e^{-\lambda t}\, dt, \tag{3}$$

where $\alpha > -1$ and $h(t)$ is smooth near 0 and doesn't grow to rapidly as $t \to \infty$.

2. Other integrals can be converted to the form (1) by a change of variable. Start with

$$J(\lambda) = \int_A^B f(s) e^{-\lambda g(s)}\, ds = e^{-\lambda g(A)} \int_A^B f(s) e^{-\lambda [g(s) - g(A)]}\, ds.$$

Set

$$t = g(s) - g(A). \tag{4}$$

If we can invert to get s as a function $s = s(t)$ then

$$ds = \frac{dt}{g'(s(t))},$$

and the integral becomes

$$J(\lambda) = e^{-\lambda g(A)} \int_0^{g(B)-g(A)} \frac{f(s(t))}{g'(s(t))} e^{-\lambda t}\, dt.$$

Since $f(s(t))/g'(s(t))$ often takes the form $t^{\alpha} h(t)$, we need to determine the asymptotics of integrals of the type (3). As usual, the main contribution as $\lambda \to \infty$ comes from the region around the minimum over of the function in the exponential. In this case the function is t, so we only have consider the integral over an interval $[0, \varepsilon]$ for some small, positive ε.

Suppose now that $h(t)$ has a Taylor series that converges uniformly over $[0, \varepsilon]$. Then for $\lambda \gg 1$,

$$\begin{aligned}
J(\lambda) &\approx \int_0^{\varepsilon} t^{\alpha} h(t) e^{-\lambda t}\, dt \\
&= \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \int_0^{\varepsilon} t^{n+\alpha} e^{-\lambda t}\, dt \\
&\approx \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \int_0^{\infty} t^{n+\alpha} e^{-\lambda t}\, dt \\
&= \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!\, \lambda^{n+1+\alpha}} \int_0^{\infty} \tau^{n+\alpha} e^{-\tau}\, d\tau \\
&= \sum_{n=0}^{\infty} \frac{h^{(n)}(0) \Gamma(n+1+\alpha)}{n!\, \lambda^{n+1+\alpha}}.
\end{aligned}$$

Thus we have the asymptotic expansion

$$J(\lambda) \sim \sum_{n=0}^{\infty} \frac{h^{(n)}(0) \Gamma(n+1+\alpha)}{n!\, \lambda^{n+1+\alpha}}, \quad \text{as } \lambda \to \infty. \tag{5}$$

The statement of (5) is *Watson's lemma.*

3. **Example**: Let

$$J(\lambda) = \int_0^{\frac{\pi}{2}} e^{-\lambda \tan^2 \theta}\, d\theta. \tag{6}$$

Then

$$J(\lambda) = \int_0^{\infty} \frac{1}{\sqrt{t}} \frac{1}{2(1+t)} e^{-\lambda t}\, dt,$$

where $t = \tan^2 \theta$. Here $\alpha = -\frac{1}{2}$ and

$$h(t) = \frac{1}{2(1+t)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n t^n, \tag{7}$$

for $|t| < 1$. By (7) and Taylor's formula,

$$\frac{h^n(0)}{n!} = \frac{1}{2} (-1)^n.$$

Plug this into (5) to get

$$J(\lambda) \sim \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\frac{1}{2})}{\lambda^{n+\frac{1}{2}}}, \quad \text{as } \lambda \to \infty.$$

Suppose we want the asymptotic behavior of $J(\lambda)$ up to $O\left(\lambda^{-\frac{5}{2}}\right)$. We know that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad \text{and} \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

It follows from (5) that

$$J(\lambda) \sim \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} - \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} + \frac{3}{8}\sqrt{\frac{\pi}{\lambda^5}}, \quad \text{as } \lambda \to \infty.$$

4. We could apply the method of Laplace directly to (6) (with $g(\theta) = \tan^2\theta$ and $f(t) \equiv 1$) to obtain the leading order behavior

$$J(\lambda) \sim \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}, \quad \text{as } \lambda \to \infty.$$