Integral Asymptotics: Laplace’s Method

1. We’ll use Laplace’s method to determine the leading-order behavior of the integral

\[ I(\lambda) = \int_a^b f(t) e^{-\lambda g(t)} \, dt, \]  

(1)

as \( \lambda \to \infty \). We’ll assume without further comment that \( I(\lambda) \) converges for \( \lambda \) sufficiently large, that \( f \) and \( g \) are smooth enough near to be replaced by local Taylor approximations of appropriate degree.

2. We’ll first consider the case in which \( g \) assumes a strict minimum over \([a, b]\) at an interior critical point \( c \). Assume that

- \( g'(c) = 0 \),
- \( g''(c) > 0 \),
- \( f(c) \neq 0 \).

We can rewrite the integral as

\[ I(\lambda) = e^{-\lambda g(c)} \int_a^b f(t) e^{-\lambda [g(t) - g(c)]} \, dt. \]  

(2)

The main idea is this: For \( \lambda \gg 1 \), the main contribution to the integral comes from a small neighborhood of \( c \). Thus, for \( \lambda \gg 1 \),

\[ I(\lambda) \approx e^{\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{-\lambda [g(t) - g(c)]} \, dt \]

\[ \approx e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\lambda [g(t) - g(c)]} \, dt \]

\[ \approx e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\lambda g'(c)(t-c) + \frac{1}{2} g''(c)(t-c)^2} \, dt \]

\[ = e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\lambda \frac{1}{2} g''(c)(t-c)^2} \, dt \]

\[ \approx e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{-\frac{1}{2} g''(c) s^2} \, ds \]

\[ = e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}}. \]
Thus, to leading order,

\[ I(\lambda) \sim e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}} \quad \text{as } \lambda \to \infty. \]  

(3)

Here, the symbol “\(\sim\)” means that the right-hand side is the first term in an asymptotic expansion of the left-hand side.

3. If \( g \) has its minimum over \([a, b]\) at an endpoint (say, \( t = a \)) with \( g'(a) = 0, g''(a) > 0 \), then analysis similar to the foregoing yields

\[ I(\lambda) \sim e^{-\lambda g(a)} f(a) \sqrt{\frac{\pi}{2\lambda g''(a)}} \quad \text{as } \lambda \to \infty, \]  

(4)

with the obvious modification when \( t = b \).

4. **Example:** We can use the method of Laplace to determine the leading order behavior of

\[ I(\lambda) = \int_{-1}^{1} \frac{\sin t}{t} e^{-\lambda \cosh t} dt. \]

Let \( g(t) = \cosh t \) and \( f(t) = \sin t/t \). The function \( g \) assumes a strict minimum over \([-1, 1]\) at the interior point \( t = 0 \), with \( g'(0) = 0 \) and \( g''(0) = 1 \). And since \( f(0) = 1 \), we have by (3),

\[ I(\lambda) \sim e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}} \quad \text{as } \lambda \to \infty. \]  

(5)

5. There are three ideas behind Laplace’s method. These are

a. For \( \lambda \gg 1 \), the main contribution to \( I(\lambda) \) comes from a small region of the minimizer \( t = c \). We can thus replace an integral over \([a, b]\) with an integral over \((c - \varepsilon, c + \varepsilon)\). (Or over \([a, a + \varepsilon]\) or \((b - \varepsilon, b]\) as the case may be).

b. In the small neighborhood of the minimizer, we can approximate \( f(t) \) and \( g(t) \) with Taylor polynomials.

c. We may extend the interval of integration to include any region that only contributes higher-order terms to \( I(\lambda) \) as \( \lambda \to \infty \).

6. **Note:** Formulas (3) and (4) are valid for infinite and semi-infinite intervals of integration, provided \( I(\lambda) \) converge for \( \lambda \) large.

7. **The Gamma Function:** For \( x > 0 \) we define the gamma function

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \]
It isn’t hard to show that
\[ \Gamma(1) = 1 \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \]

To prove the second claim we make the change of variable \( t^{\frac{1}{2}} = u \) and use the fact that
\[ \int_0^\infty e^{-au^2} \, du = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \]
for \( a > 0 \).

8. Integration by parts yields
\[ \Gamma(x + 1) = x \Gamma(x), \quad (6) \]
for any \( x > 0 \). Thus \( \Gamma(2) = \Gamma(1) = 1 \), \( \Gamma(3) = 2 \Gamma(2) = 2 \times 1 \), \( \Gamma(4) = 3 \Gamma(3) = 3 \times 2 \times 1 \), etc.

In general,
\[ \Gamma(n + 1) = n!, \]
for any nonnegative integer \( n \). Thus (6) tells us that the gamma function is a continuous generalization of the factorial function.

9. Example: Derive Stirling’s approximation:
\[ n! \sim \sqrt{2\pi n} \frac{n^\frac{1}{2}}{e} e^{-n}, \quad \text{as} \quad n \to \infty. \]

\[ \Gamma(x + 1) = \int_0^\infty t^x e^{-t} \, dt \]
\[ = \int_0^\infty e^{x \ln t} e^{-t} \, dt \]
\[ = \int_0^\infty e^{-x \left( \frac{1}{x} - \ln t \right)} \, dt \quad \text{(Set} \quad t = xz.) \]
\[ = x \int_0^\infty e^{-x \left( z - \ln xz \right)} \, dz \]
\[ = xe^{x \ln x} \int_0^\infty e^{-x \left( z - \ln z \right)} \, dz \]
\[ = xe^{x \ln x} \int_0^\infty e^{-x \left( z - \ln z \right)} \, dz. \quad (7) \]

Apply Laplace’s method to the integral in (7) with \( f(z) \equiv 1 \) and \( g(z) = z - \ln z \). We see that \( g \) has a strict minimum over \((0, \infty)\) at \( z = 1 \), with \( g(1) = 1 \), \( g'(1) = 0 \) and \( g''(1) = 1 \). Thus,
\[ \int_0^\infty e^{-x \left( z - \ln z \right)} \, dz \sim \sqrt{\frac{2\pi}{x}} e^{-x}, \quad \text{as} \quad x \to \infty. \]
and so
\[ \Gamma(x + 1) \sim x^x e^{-x} = \sqrt{\frac{2\pi}{x}} x^{x+\frac{1}{2}} e^{-x} \quad \text{as } x \to \infty. \]

Now set \( x = n \) and use the fact that \( \Gamma(n + 1) = n! \).

**10. Higher-Order Asymptotics:** With the gamma function, we can find higher-order terms in an asymptotic expansion. Suppose for example that \( g(t) \) assumes a strict minimum over \([a, b]\) at an interior point \( c \), that \( g'(c) = 0 \), \( g''(c) \neq 0 \), \( f(c) = 0 \) and \( f''(c) \neq 0 \). Then for \( \lambda \gg 1 \),

\[
I(\lambda) \approx e^{\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{-\lambda[g(t)-g(c)]} \, dt \\
\approx e^{-\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} \left[ f'(c)(t-c) + \frac{f''(c)}{2}(t-c)^2 \right] e^{-\frac{\lambda}{2}g''(c)(t-c)^2} \, dt \\
= e^{-\lambda g(c)} \int_{-\infty}^{\infty} \left[ f'(c)s + \frac{f''(c)}{2}s^2 \right] e^{-\frac{\lambda}{2}g''(c)s^2} \, ds \\
= e^{-\lambda g(c)} \frac{f''(c)}{2} \int_{-\infty}^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2} \, ds \\
= f''(c) e^{-\lambda g(c)} \int_{0}^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2} \, ds.
\]

Make the change of variable
\[ \frac{\lambda}{2}g''(c)s^2 = u, \]
and use properties of the gamma function to show that
\[
\int_{0}^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2} \, ds = \sqrt{\frac{\pi}{(\lambda g''(c))^3}}.
\]

Thus, to leading order,
\[ I(\lambda) \sim f''(c)e^{-\lambda g(c)} \sqrt{\frac{\pi}{(\lambda g''(c))^3}}, \quad \text{as } \lambda \to \infty. \]