Heisenberg's Inequality

1. Physicists generally like to take the complex conjugate of the first argument in the inner product. Hence, in this set of notes, the $L^2(\mathbf{R})$ inner product is

$$\langle f\,, g\rangle = \int \bar{f}(x) g(x)\,dx.$$

We have the Schwarz inequality

$$|\langle f\,, g\rangle| \le \|f\|\,\|g\|, \tag{1}$$

and the Plancherel identity,

$$\langle f\,, g\rangle = \langle \hat{f}\,, \hat{g}\rangle \tag{2}$$

Setting $g = f$ in (2), we get

$$\int |f(x)|^2\,dx = \int |\hat{f}(k)|^2\,dk. \tag{3}$$

2. Let ψ be the wave function, A the position operator and B the momentum:

$$\int |\psi(x)|^2\,dx = 1, \tag{4}$$

$$Af(x) = xf(x) \quad \text{and} \quad Bf(x) = (2\pi i)^{-1} f'(x). \tag{5}$$

The expected (average) value of the position is

$$E(A) = \langle \psi\,, A\psi\rangle = \int x|\psi(x)|^2\,dx. \tag{6}$$

The average value of the momentum is

$$E(B) = \langle \psi\,, B\psi\rangle = \langle \hat{\psi}\,, \widehat{B\psi}\rangle = \int k|\hat{\psi}(k)|^2\,dk. \tag{7}$$

The variance of the position is

$$\mathrm{Var}(A) = E[A - E(A)]^2 = \int [x - E(A)]^2 |\psi(x)|^2\,dx.$$

The variance of the momentum is

$$\mathrm{Var}(B) = E[B - E(B)]^2 = \int [k - E(B)]^2 |\hat{\psi}(k)|^2\,dk.$$

Heisenberg's invariance principle (in mathematical form) asserts that

$$\mathrm{Var}(A) \times \mathrm{Var}(B) \geq C > 0. \tag{8}$$

3. Before proving (8) we'll simplify the variances. Clearly,

$$\mathrm{Var}(A) = \int x^2 \left|\psi_1(x)\right|^2 \, dx, \tag{9}$$

where

$$\psi_1(x) = \psi(x + E(A)).$$

And since

$$\left|\hat{\psi}(k)\right| = \left|\hat{\psi}_1(k)\right|,$$

$$\mathrm{Var}(B) = \int [k - E(B)]^2 |\hat{\psi}_1(k)|^2 \, dk = \int k^2 |\hat{\psi}_1(k + E(B))|^2 \, dk. \tag{10}$$

Now set

$$\hat{\psi}_2(k) = \hat{\psi}_1(k + E(B)).$$

Thus,

$$\mathrm{Var}(B) = \int k^2 |\hat{\psi}_2(k)|^2 \, dk, \tag{11}$$

and

$$|\psi_1(x)| = |\psi_2(x)|.$$

Consequently,

$$\mathrm{Var}(A) = \int x^2 |\psi_2(x)|^2 \, dx. \tag{12}$$

So by (8), (12) and (11), Heisenberg's principle asserts that

$$\int x^2 |\psi_2(x)|^2 \, dx \times \int k^2 |\hat{\psi}_2(k)|^2 \, dk \geq C > 0.$$

4. We'll also need the inequality,

$$\begin{aligned} \left|x\bar{f}(x)f'(x)\right| &\geq x \,\mathrm{Re}\, \bar{f}(x)f'(x) \\ &= \frac{x}{2}\left[\bar{f}(x)f'(x) + f(x)\bar{f}'(x)\right] \\ &= \frac{x}{2}\frac{d}{dx}\bar{f}(x)f(x) \\ &= \frac{x}{2}\frac{d}{dx}|f(x)|^2. \end{aligned} \tag{13}$$

5. Weyl's proof of Heisenberg's Inequality: Let f be in the Schwarz class. Then

$$\int x^2|f(x)|^2\,dx \times \int k^2|\hat{f}(k)|\,dk \geq \frac{1}{16\pi^2}\|f\|^4. \tag{14}$$

$$\begin{aligned}
4\pi^2 \int x^2|f(x)|^2\,dx \times \int k^2|\hat{f}(k)|^2\,dk &= \int x^2|f(x)|^2\,dx \times \int |2\pi i k\hat{f}(k)|^2\,dk \\
&= \int x^2|f(x)|^2\,dx \times \int |\widehat{f'}(k)|^2\,dk \\
&= \int x^2|f(x)|^2\,dx \times \int |f'(x)|^2\,dk \\
&= \|xf\|^2\,\|f'\|^2 \\
&\geq \left[\int |xf^*(x)f'(x)|\,dx\right]^2 \\
&\geq \frac{1}{4}\left[\int x\frac{d}{dx}|f(x)|^2\,dx\right]^2 \\
&= \frac{1}{4}\left[\int |f(x)|^2\,dx\right]^2 \\
&= \frac{1}{4}\|f\|^4.
\end{aligned} \tag{15}$$

Now set $f = \psi_2$ so that

$$\|f\|^2 = \|\psi_2\|^2 = \|\psi_1\|^2 = \|\hat{\psi}_1\|^2 = \|\hat{\psi}\|^2 = \|\psi\|^2 = 1,$$

and the result follows.