The Fourier Transform 2

1. In class we used eigenfunction expansions to prove the Fourier inversion theorem and Plancherel's identity for "nice" functions of compact support on $\mathbf{R}$. You can also use approximate identities to establish these results. Here are some proofs (with a little hand-waving).

2. Proposition: If $u(x)$ decays rapidly as $|x| \to \infty$, and $\hat{u}(\xi)$ as $|\xi| \to \infty$, then

$$u(x) = \int \hat{u}(\xi) e^{2\pi i \xi \cdot x}\, d\xi. \tag{1}$$

Proof: Let $\varepsilon > 0$. Then

$$\begin{aligned}
\int \hat{u}(\xi) e^{2\pi i \xi \cdot x}\, d\xi &= \int \left[\int u(y) e^{-2\pi i \xi \cdot y}\, dy \right] e^{2\pi i \xi \cdot x}\, d\xi \\
&= \lim_{\varepsilon \to 0} \int u(y) \left[\int e^{2\pi i \xi \cdot (x-y)} e^{-4\varepsilon \pi^2 |\xi|^2}\, d\xi \right] dy \\
&= \lim_{\varepsilon \to 0} \int G_\varepsilon(x-y) u(y)\, dy \\
&= \int \delta(x-y) u(y)\, dy \\
&= u(x).
\end{aligned}$$

3. Note: You have to take some care with the interpretation of (1). In what sense does equality hold? If u decays rapidly as $|x| \to \infty$ as is continuous, then equality in (1) is pointwise. If u is in $L^2(\mathbf{R}^n)$ but not necessarily continuous, then equality holds in the sense of L^2:

$$\int |u(x) - (\hat{u})\check{}(x)|^2\, dx = 0.$$

4. Proposition: For u and v in $L^2(\mathbf{R}^n)$,

$$\langle \hat{u}\,, \hat{v} \rangle = \langle u\,, v \rangle \tag{2}$$

and hence

$$\|\hat{u}\|_2 = \|u\|_2. \tag{3}$$

These are versions of the Plancherel identity.

Proof: We'll use the fact that the Gauss kernel is an approximate identity.

$$
\begin{aligned}
\langle \hat{u}\,, \hat{v}\rangle &= \int \hat{u}(\xi)\bar{\hat{v}}(\xi)\,d\xi \\
&= \int \left[\int u(x)e^{-2\pi i\xi\cdot x}\,dx\right]\left[\int \bar{v}(y)e^{-2\pi i\xi\cdot y}\,dy\right]\,d\xi \\
&= \lim_{\varepsilon\to 0}\int u(x)\int \bar{v}(y)\int e^{-4\varepsilon\pi^2|\xi|^2}e^{2\pi i\xi\cdot(x-y)}\,d\xi\,dy\,dx \\
&= \lim_{\varepsilon\to 0}\int u(x)\int \bar{v}(y)G_\varepsilon(y-x)\,dy\,dx \\
&= \int u(x)\int \bar{v}(y)\delta(y-x)\,dy\,dx \\
&= \int u(x)\bar{v}(x)\,dx \\
&= \langle u\,, v\rangle.
\end{aligned}
$$

To get (3) from (2), just take $v = u$.

5. An operator that preserves inner products is called unitary. Since

$$\langle \mathcal{F}u\,, \mathcal{F}v\rangle = \langle u\,, v\rangle$$

the Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

6. Let $\varepsilon > 0$ and $\xi \in \mathbf{R}$. Find the function $P_\varepsilon(x)$ whose Fourier transform is

$$\hat{P}_\varepsilon(\xi) = e^{-2\pi\varepsilon|\xi|}.$$

By the inversion theorem,

$$P_\varepsilon(x) = \int \hat{P}_\varepsilon(\xi)e^{2\pi i\xi x}\,d\xi = \frac{1}{\pi}\,\frac{\varepsilon}{\varepsilon^2 + x^2}.$$

The function P_ε is called the Possion kernel (for the upper half-plane). It is also an approximate identity as $\varepsilon \downarrow 0$.

7. The characteristic or indicator of a set A is

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \neq A. \end{cases}$$

Let $A = [-a, a]$ in $\mathbf{R}$. The function whose Fourier transform is $\chi_A(\xi)$ is

$$\begin{aligned} f(x) &= \check{\chi}_A(x) \\ &= \int \chi(\xi) e^{2\pi i \xi x}\, d\xi \\ &= \frac{e^{2\pi i a x} - e^{-2\pi i a x}}{2\pi i x} \\ &= \frac{\sin(2\pi a x)}{\pi x}. \end{aligned}$$

8. We can extend the Fourier and inverse Fourier transforms to objects like the Dirac delta function (not really a function) and to functions like $f(x) \equiv 1$, for which the Fourier integral (1) doesn't converge. By the defining property of the delta function,

$$\mathcal{F}[\delta(x-y)] = \int \delta(x-y) e^{-2\pi i \xi \cdot x}\, d\xi = e^{-2\pi i \xi \cdot y}. \tag{4}$$

If $y = 0$, this becomes

$$\hat{\delta}(x) = 1.$$

Thus,

$$\mathcal{F}^{-1}[e^{-2\pi i \xi \cdot y}] = \delta(x-y),$$

and

$$\check{1}(x) = \delta(x).$$

By the definition of the Fourier and inverse Fourier transforms,

$$\hat{f}(\xi) = \check{f}(-\xi). \tag{5}$$

And thus, formally, for fixed $y \in \mathbf{R}^n$,

$$\mathcal{F}[e^{2\pi i \xi \cdot y}] = \mathcal{F}^{-1}[e^{-2\pi i \xi \cdot y}] = \delta(x-y).$$

This sort of formal calculation can be made rigorous with the theory of tempered distributions.