1. In class we used eigenfunction expansions to prove the Fourier inversion theorem and Plancherel’s identity for “nice” functions of compact support on $\mathbb{R}$. You can also use approximate identities to establish these results. Here are some proofs (with a little hand-waving).

2. **Proposition**: If $u(x)$ decays rapidly as $|x| \to \infty$, and $\hat{u}(\xi)$ as $|\xi| \to \infty$, then

$$u(x) = \int \hat{u}(\xi) e^{2\pi i \xi \cdot x} d\xi. \quad (1)$$

**Proof**: Let $\varepsilon > 0$. Then

$$\int \hat{u}(\xi) e^{2\pi i \xi \cdot x} d\xi = \int \left[ \int u(y) e^{-2\pi i \xi \cdot y} dy \right] e^{2\pi i \xi \cdot x} d\xi$$

$$= \lim_{\varepsilon \to 0} \int u(y) \left[ \int e^{2\pi i \xi \cdot (x-y)} e^{-4\varepsilon \pi^2 |\xi|^2} d\xi \right] dy$$

$$= \lim_{\varepsilon \to 0} \int G_\varepsilon(x-y) u(y) dy$$

$$= \int \delta(x-y) u(y) dy$$

$$= u(x).$$

3. **Note**: You have to take some care with the interpretation of (1). In what sense does equality hold? If $u$ decays rapidly as $|x| \to \infty$ as is continuous, then equality in (1) is pointwise. If $u$ is in $L^2(\mathbb{R}^n)$ but not necessarily continuous, then equality holds in the sense of $L^2$:

$$\int |u(x) - (\hat{u})^{-}(x)|^2 dx = 0.$$

4. **Proposition**: For $u$ and $v$ in $L^2(\mathbb{R}^n)$,

$$\langle \hat{u}, \hat{v} \rangle = \langle u, v \rangle \quad (2)$$

and hence

$$\|\hat{u}\|_2 = \|u\|_2. \quad (3)$$

These are versions of the Plancherel identity.
Proof: We’ll use the fact that the Gauss kernel is an approximate identity.

\[ \langle \hat{u}, \hat{v} \rangle = \int \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi \]

\[ = \int \left[ \int u(x) e^{-2\pi i \xi \cdot x} dx \right] \left[ \int \bar{v}(y) e^{-2\pi i \xi \cdot y} dy \right] d\xi \]

\[ = \lim_{\varepsilon \to 0} \int u(x) \int \bar{v}(y) \int e^{-4\varepsilon |\xi|^2} e^{2\pi i \xi \cdot (x - y)} \, d\xi \, dy \, dx \]

\[ = \lim_{\varepsilon \to 0} \int u(x) \int \bar{v}(y) G_\varepsilon(y - x) \, dy \, dx \]

\[ = \int u(x) \int \bar{v}(y) \delta(y - x) \, dy \, dx \]

\[ = \int u(x)\bar{v}(x) \, dx \]

\[ = \langle u, v \rangle. \]

To get (3) from (2), just take \( v = u \).

**5.** An operator that preserves inner products is called unitary. Since

\[ \langle \mathcal{F}u, \mathcal{F}v \rangle = \langle u, v \rangle \]

the Fourier transform is a unitary operator on \( L^2(\mathbb{R}^n) \).

**6.** Let \( \varepsilon > 0 \) and \( \xi \in \mathbb{R} \). Find the function \( P_\varepsilon(x) \) whose Fourier transform is

\[ \hat{P}_\varepsilon(\xi) = e^{-2\pi \varepsilon |\xi|}. \]

By the inversion theorem,

\[ P_\varepsilon(x) = \int \hat{P}_\varepsilon(\xi) e^{2\pi i \xi x} \, d\xi = \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + x^2}. \]

The function \( P_\varepsilon \) is called the Possion kernel (for the upper half-plane). It is also an approximate identity as \( \varepsilon \downarrow 0 \).

**7.** The characteristic or indicator of a set \( A \) is

\[ \chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \not\in A. \end{cases} \]
Let $A = [-a, a]$ in $\mathbb{R}$. The function whose Fourier transform is $\chi_A(\xi)$ is

$$f(x) = \hat{\chi}_A(x) = \int \chi(\xi)e^{2\pi i \xi x} d\xi = \frac{e^{2\pi i a x} - e^{-2\pi i a x}}{2\pi i x} = \frac{\sin(2\pi a x)}{\pi x}.$$ 

8. We can extend the Fourier and inverse Fourier transforms to objects like the Dirac delta function (not really a function) and to functions like $f(x) \equiv 1$, for which the Fourier integral (1) doesn’t converge. By the defining property of the delta function,

$$\mathcal{F}[\delta(x-y)] = \int \delta(x-y)e^{-2\pi i \xi \cdot x} d\xi = e^{-2\pi i \xi \cdot y}. \quad (4)$$

If $y = 0$, this becomes

$$\hat{\delta}(x) = 1.$$ 

Thus,

$$\mathcal{F}^{-1}[e^{-2\pi i \xi \cdot y}] = \delta(x-y),$$ 

and

$$\check{1}(x) = \delta(x).$$

By the definition of the Fourier and inverse Fourier transforms,

$$\hat{f}(\xi) = \check{f}(-\xi). \quad (5)$$

And thus, formally, for fixed $y \in \mathbb{R}^n$,

$$\mathcal{F}[e^{2\pi i \xi \cdot y}] = \mathcal{F}^{-1}[e^{-2\pi i \xi \cdot y}] = \delta(x-y).$$

This sort of formal calculation can be made rigorous with the theory of tempered distributions.