1. **Example:** Let $u(x,t)$ be the density of a gas in a straight, narrow, cylindrical tube of length 1. Let $f$ be the initial density. Assume that the ends of the tube are plugged. Thus $u$ satisfies the initial-boundary value problem with “no-flux” boundary conditions:

$$
(P_0) \begin{cases}
  u_t - ku_{xx} = 0, & \text{for } 0 < x < 1, \ t > 0, \\
  u(x,0) = f(x), \\
  u_x(0,t) = 0, \\
  u_x(1,t) = 0.
\end{cases}
$$

We separate variables:

$$u(x,t) = T(t)X(x), \quad (1)$$

and see that

$$T'(t) = \lambda kT(t), \quad (2)$$

and

$$
(P_1) \begin{cases}
  X'' = \lambda X, & \text{for } 0 < x < 1, \\
  X'(0) = 0, \\
  X'(1) = 0,
\end{cases}
$$

for some constant $\lambda$. Problem $(P_1)$ is called an eigenvalue problem. Let

$$L = \frac{d^2}{dx^2} = D^2,$$

and define the linear boundary operators

$$B_1 X = X'(0),$$

and

$$B_2 X = X'(1).$$

With this notation, $(P_1)$ is

$$
(P_1) \begin{cases}
  LX = \lambda X, & \text{for } 0 < x < 1, \\
  B_1 X = 0, \\
  B_2 X = 0.
\end{cases}
$$

Since the first boundary condition involves $X$ only at $x = 0$, and the second, only at $x = 1$, the boundary conditions are called *separated*.

2. **Example:** Suppose now $u(x,t)$ represents the temperature of a thin, insulated, wire ring of circumference 1. Here, the spatial variable $x$ represents arclength along the
ring, measured widdershins (counterclockwise). We thus have the periodic boundary conditions
\[ u(0, t) = u(1, t), \]  
and
\[ u_x(0, t) = u_x(1, t). \]  
With the initial temperature distribution \( f \), we have the initial boundary value problem
\[
\begin{cases}
  u_t - ku_{xx} = 0, & \text{for } 0 < x < 1, \ t > 0, \\
  u(x, 0) = f(x), \\
  u(0, t) - u(1, t) = 0, \\
  u_x(0, t) - u_x(1, t) = 0.
\end{cases}
\]  
We set
\[ u(x, t) = T(t)X(x), \]
and obtain
\[ T'(t) = \lambda k T(t), \]  
and
\[
\begin{cases}
  X'' = \lambda X, & \text{for } 0 < x < 1, \\
  X(0) - X(1) = 0, \\
  X'(0) - X'(1) = 0,
\end{cases}
\]  
for some constant \( \lambda \). If we set
\[ L = D^2, \]
and define the linear boundary operators
\[ B_1 X = X(0) - v(1), \]
and
\[ B_2 X = X'(0) - v'(1), \]
\((P_3)\) becomes
\[
\begin{cases}
  LX = \lambda X, & \text{for } 0 < x < 1, \\
  B_1 X = 0, \\
  B_2 X = 0.
\end{cases}
\]  
3. We’ll say that a boundary operator \( B_i \) is of order \( k \) if it contains derivatives up to but not exceeding the \( k \)th. In the first example, the both boundary operators are of order 1. In the second, \( B_1 \) is of order 0 and \( B_1 \) of order 1.

4. Let \([a, b]\) be a finite interval. Define the second-order, linear differential operator
\[ L = a_2(x)D^2 + a_1(x)D + a_0, \]
where the $a_i$ are smooth and complex-valued, and $a_2(x) \neq 0$ on $[a, b]$. Let $B_1$ and $B_2$ be linear boundary operators of at most the first order. Consider the problem

$$
(P_4) \begin{cases}
LX = \lambda X, & \text{for } a < x < b, \\
B_1X = 0, \\
B_2X = 0.
\end{cases}
$$

5. **Note:** The function $X \equiv 0$ is a solution (called the trivial solution) to $(P_4)$. A solution $X$ that is not identically zero is called nontrivial.

6. **Note:** We haven’t been specific about the domain of $L$, that is, the class of functions $X$ on which $L$ operates. We require that

a. $X$, $X'$ and $X''$ be in $L^2[a, b]$,

b. $B_1X = B_2X = 0$.

For practical purposes, you don’t have to worry about (a). Just remember that functions in the domain of $L$ have to satisfy the boundary conditions.

7. **Linear algebraic digression:** Let $A = (a_{ij})$ be a complex, $n \times n$ matrix, and $\lambda$ a scalar. If the equation

$$AX = \lambda X, \quad (6)$$

has a solution $X \neq 0$, then $\lambda$ is an eigenvalue of $A$. Any vector $X$ satisfying (6) is an eigenvector belonging to $\lambda$. Note that $X = 0$ (i.e. the zero vector in $\mathbb{C}^n$) is an eigenvector belonging to every eigenvalue.

8. **Proposition:** The set of eigenvectors belonging to an eigenvalue $\lambda$ is a linear subspace of $\mathbb{C}^n$. (It is called the eigenspace of $\lambda$.)

9. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$. Let $A^* = (\overline{a_{ji}})$ and be the adjoint of $A$. Then

$$\langle AX, Y \rangle = \langle X, A^*Y \rangle, \quad (7)$$

for all $X$ and $Y$ in $\mathbb{C}^n$.

10. $A$ is called self-adjoint if $A = A^*$. If $A$ is self-adjoint then (7) becomes

$$\langle AX, Y \rangle = \langle X, AY \rangle, \quad (8)$$

for all $X$ and $Y$ in $\mathbb{C}^n$. This can be used to prove two important propositions:

11. **Proposition:** The eigenvalues of a self-adjoint matrix are real.

12. **Proposition:** Let $A$ be a self-adjoint matrix. If $X$ and $Y$ be eigenvectors belonging respectively to the distinct eigenvalues $\mu$ and $\lambda$, then $\langle X, Y \rangle = 0$. Thus eigenspaces of distinct eigenvalues of the self-adjoint matrix $A$ are orthogonal.
13. **Definition:** Consider \((P_4)\) with \(\lambda\) fixed.

\[
\begin{cases}
  LX = \lambda X, & \text{for } a < x < b, \\
  B_1 X = 0, \\
  B_2 X = 0.
\end{cases}
\]

If there is a nontrivial solution to this problem, then \(\lambda\) is called an eigenvalue. *Any* solution is called an eigenfunction belonging to \(\lambda\). Note that the trivial solution \(X \equiv 0\) is an eigenfunction of every eigenvalue.

14. **Proposition:** The set of eigenfunctions belonging to an eigenvalue \(\lambda\) forms a vector space. (This is called the eigenspace of \(\lambda\). It is a subspace of \(L^2[a, b]\).)

15. **Self-Adjoint Problems:** Integration by parts yields

\[
\int_a^b LX(x)\bar{Y}(x) \, dx = B(X, \bar{Y}) + \int_a^b X(x)L^*Y(x) \, dx, \tag{9}
\]

where \(B(X, \bar{Y})\) represents the boundary terms and

\[
L^*Y = (\bar{a}_2 Y)'' - (\bar{a}_1 Y)' + \bar{a}_0 Y, \tag{10}
\]

is the formal adjoint of \(L\). If \(\langle , \rangle\) is the \(L^2\) inner product on \([a, b]\), then (9) becomes

\[
\langle LX, Y \rangle = B(X, Y) + \langle X, L^*Y \rangle. \tag{11}
\]

16. If

\[
L = L^*, \quad \text{and} \quad B(X, \bar{Y}) = 0,
\]

for all \(X\) and \(Y\) in the domain of \(L\), then

\[
\langle LX, Y \rangle = \langle X, LY \rangle. \tag{12}
\]

When this is the case, the problem is called *self-adjoint*.

17. **Example:** Problems \((P_1)\) and \((P_3)\) are self-adjoint.

18. **Proposition:** The eigenvalues of a self-adjoint problem are real.

19. **Proposition:** Let \((P_4)\) be self-adjoint. and let \(\lambda \neq \mu\) be eigenvalues. If \(Y\) and \(Z\) are eigenvectors belonging to \(\lambda\) and \(\mu\) respectively, then \(\langle Y, Z \rangle = 0\). (Thus the eigenspaces of \(\lambda\) and \(\mu\) are orthogonal.)

20. If the coefficient functions \(a_i\) are *real-valued* then \(L^* = L\) if and only if

\[
LX = (a_2X')' + a_0 X.
\]

We usually set \(a_2 = -p\) and \(a_0 = q\) and write the operator in Sturm-Liouville form:

\[
LX = -(pX')' + qX.
\]