The Heat, Laplace and Poisson Equations

1. Let $u = u(x,t)$ be the density of stuff at $x \in \mathbf{R}^n$ and time t. Let J be the flux density vector. If stuff is conserved, then

$$u_t + \operatorname{div} J = 0. \tag{1}$$

If the density is changing by diffusion only, the simplest constitutive equation is

$$J = -k\nabla u, \tag{2}$$

where $k > 0$ is the diffusion coefficient. This is Fourier's or Fick's first law. Our conservation law thus becomes

$$u_t - \operatorname{div}(k\nabla u) = 0. \tag{3}$$

2. In certain cases, it is reasonable to take k to be a positive, increasing function of u. Thus $k = k(u) > 0$ and $k'(u) > 0$ for $u > 0$. The equation (3) becomes

$$u_t - \operatorname{div}(k(u)\nabla u) = 0. \tag{4}$$

In $n = 1$ space dimension, the divergence and the gradient operators reduce to the spatial derivative. We thus obtain the PDE

$$u_t - (k(u)u_x)_x = 0. \tag{5}$$

It is sometimes the case that the medium is not spatially uniform, and that $k = k(x) > 0$. Hence,

$$u_t - \operatorname{div}(k(x)\nabla u) = 0. \tag{6}$$

3. Note that

$$\operatorname{div}(\nabla u) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} \equiv \Delta u.$$

The operator

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

is called the Laplacian.

4. **The Heat equation:** In the simplest case, $k > 0$ is a constant. Our conservation law becomes

$$u_t - k\Delta u = 0. \tag{7}$$

This is the heat equation to most of the world, and Fick's second law to chemists.

5. **Laplace's equation:** Suppose that as $t \to \infty$, the density function $u(x,t)$ in (7) approaches a time-independent equilibirium $w(x) = u(x,\infty)$. The time derivative drops out, leaving

$$\Delta w = 0. \tag{8}$$

This is Laplace's equation.

6. **Poisson's equation:** The heat equation with source term $f(x)$ is

$$u_t - k\Delta u = f(x). \tag{9}$$

In this case the equilibrium density w satisfies Poisson's equation:

$$-\Delta w = g(x), \tag{10}$$

where $g(x) = k^{-1} f(x)$.

7. The same partial differential equation can arise in different settings. Consider Gauss' law from electrostatics. If $\varrho(x)$ is the charge density and $E = E(x)$ the resulting electric field, then in integral form, this is

$$\frac{1}{\varepsilon_0}\int_B \varrho\, dx = \int_Q E \cdot \nu\, dS. \tag{11}$$

The constant ε_0 is the permitivity of free space. By the divergence theorem,

$$\operatorname{div} E = \frac{1}{\varepsilon_0}\varrho. \tag{12}$$

Let φ be the field potenital:

$$E = -\nabla\varphi. \tag{13}$$

Plug this into the previous equation to obtain

$$-\Delta\varphi = \frac{1}{\varepsilon_0}\varrho, \tag{14}$$

which is Poisson's equation. In a charge-free region of space, φ satisfies Laplace's equation:

$$\Delta\varphi = 0. \tag{15}$$