Calculus of Variations 7: Hamilton's Equations

1. Let T be the kinetic energy and U the potential energy of a mechanical system. Let $q_k(t)$, $k = 1, \ldots, n$ be generalized coordinates. According to Hamilton's principle, the trajectory of $q(t) = (q_1(t), \ldots, q_n(t))$ through the configuration space $\mathcal{C}$ satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \quad k = 1, \ldots, n, \tag{1}$$

where the Lagranigian is $L = T - U$.

2. We can also characterize a mechanical system in terms of the generalized coordinates q_k *and* the generalized momenta p_k. The $2n$-dimensional set $\mathcal{P}$ of points $x = (q, p)$ is called the *phase space.* Adding n new variables doesn't seem like a step forward, but doing so will allow us to replace the system (1) of n second-order equations with one comprising $2n$ first-order equations. These are called *Hamilton's equations.* They determine the trajectory of the phase point $(q(t), p(t))$ through $\mathcal{P}$, in other words, the behavior of the mechanical system.

3. The generalized momenta are

$$p_k = L_{\dot{q}_k}(t, q, \dot{q}), \quad k = 1, \ldots, n. \tag{2}$$

Think of (2) as a system of n equations in n variables $\dot{q}_1, \ldots, \dot{q}_n$. We solve the system for

$$\dot{q}_k = \dot{q}_k(t, q, p), \quad k = 1, \ldots, n. \tag{3}$$

In vector form, this is

$$\dot{q} = \dot{q}(t, q, p). \tag{4}$$

The Hamiltonian is

$$\begin{aligned} H &= -L(t, q, \dot{q}) + \sum_{k=1}^{n} \dot{q}_k L_{\dot{q}_k}(t, q, \dot{q}) \\ &= -L(t, q, \dot{q}(t, q, p)) + \sum_{k=1}^{n} p_k \dot{q}_k(t, q, p). \end{aligned} \tag{5}$$

(5) gives the Hamiltonian as a function of t, p and q. The partial derivatives of H with respect to the q_j and p_j are

$$\begin{aligned} \frac{\partial H}{\partial p_j} &= -\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial p_j} + \sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial p_j} + \dot{q}_j \\ &= -\sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial p_j} + \sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial p_j} + \dot{q}_j \\ &= \dot{q}_j, \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
\frac{\partial H}{\partial q_j} &= -\frac{\partial L}{\partial q_j} - \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial q_j} + \sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial q_j} \\
&= -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial q_j} + \sum_{k=1}^{n} p_k \frac{\partial \dot{q}_k}{\partial q_j} \\
&= -\dot{p}_j. \qquad (7)
\end{aligned}
$$

Thus,

$$
\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \text{for} \quad j = 1, \dots, n. \qquad (8)
$$

These are Hamilton's equations.

4. **Example**: Consider a pendulum with string of length l and negligible mass and and a bob of mass m. The generalized coordinate is the angle θ made by the string and the pendulum. The kinetic and potential energies are

$$
T = \frac{1}{2} m (l\dot{\theta})^2 \quad \text{and} \quad U = mgl(1 - \cos\theta).
$$

Hence the The Lagrangian is

$$
L(\theta, \dot{\theta}) = \frac{1}{2} m (l\dot{\theta})^2 - mgl(1 - \cos\theta). \qquad (9)
$$

By definition, the generalized momentum is

$$
p = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta},
$$

so that

$$
\dot{\theta} = \dot{\theta}(p) = \frac{1}{ml^2}\, p.
$$

Finally, by (5), the Hamiltonian is

$$
\begin{aligned}
H &= -L(\theta, \dot{\theta}) + p\dot{\theta} \\
&= -\frac{1}{2} m (l\dot{\theta})^2 + mgl(1 - \cos\theta) + p\dot{\theta} \\
&= -\frac{1}{2ml^2}\, p^2 + mgl(1 - \cos\theta) + \frac{1}{ml^2}\, p^2 \\
&= \frac{1}{2ml^2}\, p^2 + mgl(1 - \cos\theta). \qquad (10)
\end{aligned}
$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{1}{m\Lambda^2}\, p,$$

and

$$\dot{p} = -\frac{\partial H}{\partial \theta} = -mg\Lambda \sin\theta.$$

Recall the equation of motion in Lagrangian form, derived from Hamilton's principle in the previous set of of notes:

$$L_\theta - \frac{d}{dt} L_{\dot{\theta}} = -mgl\sin\theta - ml^2\ddot{\theta} = 0,$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

5. **Example**: Consider a body of mass m moving frictionlessly at the end of a spring. Let q be the displacement from equilibrium of the mass. By Hooke's law, the restoring force of the spring is roughly $-kq$ for small amplitude motion. Hence the potential is

$$U = \frac{k}{2}q^2.$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{q}^2,$$

and the Lagrangian,

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{k}{2}q^2. \tag{11}$$

Therefore,

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \tag{12}$$

is the generalized momentum. In this case, it is simply the standard momentum. The Hamiltonian is thus

$$\begin{aligned} H &= -L + p\dot{q} \\ &= -\frac{1}{2}m\dot{q}^2 + \frac{k}{2}q^2 + p\dot{q} \\ &= -\frac{1}{2m}p^2 + \frac{k}{2}q^2 + \frac{1}{m}p^2 \\ &= \frac{1}{2m}p^2 + \frac{k}{2}q^2. \end{aligned} \tag{13}$$

So Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m},$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq.$$

You can get the equations of motion in Lagrangian form from (11):

$$L_q - \frac{d}{dt} L_{\dot{q}} = -(kq - m\ddot{q}) = 0,$$

or

$$\ddot{q} + \frac{k}{m} q = 0,$$

which is the familiar simple harmonic oscillator.

6. The derivative of H along the path traced by $(q(t), p(t))$ in the phase space is

$$\begin{aligned} \frac{d}{dt} H(t, q(t), p(t)) &= \frac{\partial H}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) \\ &= \frac{\partial H}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \\ &= \frac{\partial H}{\partial t}. \end{aligned} \tag{14}$$

We'll draw an important conclusion from (14).

7. In this and the previous set of notes, we've seen three examples of kinetic energy. There was

$$T = T(\dot{\theta}) = \frac{1}{2} m \Lambda^2 \dot{\theta}^2, \tag{15}$$

for the pendulum,

$$T = T(r, \dot{r}, \dot{\theta}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2). \tag{16}$$

for the mass in the gravitational field, and

$$T(\dot{q}) = \frac{1}{2} m \dot{q}^2. \tag{17}$$

In each case, T was a quadratic form in the generalized velocities. The coefficients of the quadratic form were functions of the generalized coordinates. Following these examples, we will take the kinetic energy to have the form

$$T(q, \dot{q}) = \sum_{i,j=1}^{n} a_{ij}(q) \dot{q}_i \dot{q}_j, \tag{18}$$

where $a_{ij}(q) = a_{ji}(q)$. In (16), for example, $q = (r, \theta)$, and the coefficients are

$$a_{11}(q) = \frac{1}{2}m, \quad a_{12}(q) = a_{21}(q) = 0, \quad \text{and} \quad a_{22}(q) = \frac{1}{2}mr^2. \tag{19}$$

We will also assume that the potential U *does not* depend on $\dot{q}$. Under these assumptions, the Lagrangian has the form

$$L(t, q, \dot{q}) = \sum_{i,j=1}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j - U(q, t). \tag{20}$$

8. Under the assumptions made in the previous paragraph,

$$\frac{\partial L}{\partial \dot{q}_j} = 2\sum_{i=1}^{n} a_{ij}(q)\dot{q}_i.$$

Here, we have used the fact that $a_{ij} = a_{ji}$. Therefore,

$$\begin{aligned} H &= -L + \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \\ &= -L + 2\sum_{i,j=1}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j, \\ &= -T + U + 2T \\ &= T + U. \end{aligned} \tag{21}$$

So if L has the form (20), then H is the total energy of the mechanical system.

9. We can draw certain important conclusions from Hamilton's equations. We already know that if the Lagrangian does not depend on time, then the Hamiltonian is a first integral. This fact, along with (21) proves that the total energy is conserved when $L = L(q, \dot{q})$. That conclusion can also be drawn straight from Hamilton's equations: If $H = H(q, p)$, then (14) reduces to

$$\frac{d}{dt}H(q(t), p(t)) = 0. \tag{22}$$

Thus the total energy is conserved.