Calculus of Variations 4: Several Functions of a Single Variable

1. Let $x_1(t), \ldots, x_n(t)$ be real-valued functions defined on an interval $a \le t \le b$. Let $x : [a, b] \mapsto \mathbf{R}^n$ by

$$x(t) = (x_1(t), \ldots, x_n(t)). \tag{1}$$

The function x is *vector-valued,* or specifically, $\mathbf{R}^n$*-valued.* You can think of $x(t)$ as tracing a curve in $\mathbf{R}^n$ as t increases from a to b. We will refer to both the function and the curve as x.

2. The derivative of x at t is the vector

$$\dot{x}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t)). \tag{2}$$

The vector $\dot{x}(t)$ is tangent to the curve traced by x at the point $x(t)$. If $x(t)$ is the position of a moving particle, then $\dot{x}(t)$ is velocity of the particle.

3. We consider functionals of the form

$$J(x) = \int_a^b L(t, x(t), \dot{x}(t))\, dt. \tag{3}$$

The Lagrangian L is a function of $2n + 1$ arguments. We

a. The functions x in the domain $\mathcal{D}$ of J should be smooth. This means that the components x_k have as many continuous derivatives as we require.

b. We impose the boundary conditions on functions in $\mathcal{D}$,

$$x(a) = \alpha \quad \text{and} \quad x(b) = \beta, \tag{4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. A function x satisfying (4) traces a curve in $\mathbf{R}^n$ from α to β.

c. Admissible variations are smooth functions $g(t) = (g_1(t), \ldots, g_n(t))$ which vanish at a and b. Thus

$$g(a) = g(b) = (0, , \ldots, 0) = 0 \in \mathbf{R}^n. \tag{5}$$

As usual, the class of admissible variations is denoted $\mathcal{A}$. For $\varepsilon \ll 1$, $x + \varepsilon g$ is a curve from α to β that is "close" to x.

4. The Gâteaux variation of J at $x \in \mathcal{D}$ in the direction $h \in \mathcal{A}$ is

$$\begin{aligned} \delta J(x, g) &= \frac{d}{d\varepsilon}\, J(x + \varepsilon g)\big|_{\varepsilon=0} \\ &= \int_a^b \sum_{k=1}^n \left(\frac{\partial L}{\partial x_k}\, g_k + \frac{\partial L}{\partial \dot{x}_k}\, \dot{g}_k \right) dt \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \sum_{k=1}^n \left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} \right) g_k \, dt + \sum_{k=1}^n \frac{\partial L}{\partial \dot{x}_k} \, g_k \Big|_a^b \\
&= \int_a^b \sum_{k=1}^n \left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} \right) g_k \, dt. \qquad (6)
\end{aligned}
$$

By (6), $x \in \mathcal{D}$ is an extremal of J if it satisfies the *system* of n Euler equations,

$$
\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = 0, \quad k = 1, \dots, n. \qquad (7)
$$

5. The Hamiltonian is

$$
H = -L(t, x, \dot{x}) + \sum_{k=1}^n \dot{x} \, \frac{\partial L}{\partial \dot{x}_k}(t, x, \dot{x}). \qquad (8)
$$

- **a**. When $L = L(x, \dot{x})$, the Hamiltonian is a first integral of the system of Euler equations. This means that

$$
H = -L(x, \dot{x}) + \sum_{k=1}^n \dot{x} \, \frac{\partial L}{\partial \dot{x}_k}(x, \dot{x}) = \text{constant},
$$

  if x satisfies (7).
- **b**. If $L = L(t, \dot{x})$, then the n functions $\partial L / \partial \dot{x}_k$, $k = 1, \dots, n$ are first intgrals of the system (7).
- **c**. If $L = L(t, x)$, then (7) reduces to the system of algebraic equations

$$
\frac{\partial L}{\partial x_k} = 0, \quad k = 1, \dots, n.
$$