1. A fluid is flowing in a region in $\mathbb{R}^3$. Let $v(x, t)$ and $\rho(x, t)$ be the velocity vector and mass density function at $x \in \mathbb{R}^3$ and time $t$. Note that $v$ is a time dependent vector field on $\mathbb{R}^3$ and that $\rho$ is a scalar-valued function. Let $\partial B$ be a smooth, closed surface bounding a volume $B$.

2. Let $M = M(t)$ be the mass of fluid in $B$ at time $t$. Clearly,

$$\frac{dM}{dt} = \{\text{Net flux of fluid out of } B \text{ through } \partial B\} - \{\text{Net rate at which fluid is created in } B\}.$$ (1)

When computing the net flux out of $B$, we count as negative flow into $B$ and positive flow out. Similarly, the net rate at which fluid is produced is the rate of fluid creation minus the rate of fluid destruction.

3. The mass in $B$ is

$$M = \int_B \rho \, dx.$$ (2)

Hence

$$\frac{dM}{dt} = \frac{d}{dt} \int_B \rho \, dx.$$ (3)

4. Let $P$ be an infinitesimally small patch of $\partial B$ with area $dS$. We take $P$ so small as to be virtually flat. Thus $\nu$ is constant over $P$. By the same token, we can take $v$ and $\rho$ to be constant over $P$ between times $t$ and $t + dt$. Let $\theta$ be the angle between $\nu$ and $v$. Over the time period $[t, t + dt]$, the net amount of fluid flowing through $P$ is

$$dm = \rho |v| dt \cos \theta \, dS = \rho v \cdot \nu \, dS dt.$$ 

Thus the net rate of fluid flow (the flux) through $P$ is

$$\frac{dm}{dt} = \rho v \cdot \nu \, dS.$$ 

Integrate the above expression over $B$ to obtain

$$\{\text{Net flux of fluid out of } B \text{ through } \partial B\} = \int_{\partial B} \rho v \cdot \nu \, dS.$$ (4)

5. Let $f(x, t)$ be the net rate per unit volume at which fluid is produced at the point $x$ and time $t$. Thus

$$\{\text{Net rate at which fluid is created in } B\} = \int_B f \, dx.$$ (5)
6. Plug (3), (4) and (5) into the balance law (1):

\[ \frac{d}{dt} \int_B \rho \, dx + \int_{\partial B} \rho v \cdot \nu \, dS = \int_B f \, dx. \] (6)

This is the balance law in its most general form. If \( \rho \) and \( v \) are smooth, we can move the time derivative into the first integral and apply the divergence theorem to the second, yielding

\[ \frac{d}{dt} \int_B \rho \, dx = \int_B \rho_t \, dx, \]

and

\[ \int_{\partial B} \rho v \cdot \nu \, dS = \int_B \text{div} (\rho v) \, dx. \]

We can thus rewrite the balance law as

\[ \int_B [\rho_t + \text{div} (\rho v)] \, dx = \int_B f \, dx. \] (7)

Equations (6) and (7) are the integral forms of the balance law. If (7) holds for every bounded volume \( B \), then it must hold pointwise. Thus,

\[ \rho_t + \text{div} (\rho v) = f, \] (8)

which is the balance law in differential form. Finally, if fluid is neither created nor destroyed, then \( f \equiv 0 \), and (8) becomes the conservation law

\[ \rho_t + \text{div} (\rho v) = 0. \] (9)

(9) is usually called the continuity equation.

7. The mass flux of the fluid through \( \partial B \) is

\[ \int_{\partial B} \rho v \cdot \nu \, dS. \]

The vector \( \rho v \) is called the flux density vector, or simply the flux vector. In general, if the flux of some quantity (call it “stuff”) across a patch of area \( dS \) is \( J \cdot \nu \, dS \), then \( J \) is the flux density vector for the flow of stuff. Note that \( J \) has dimensions

\[ [J] = [\text{stuff}] \times \text{area}^{-1} \times \text{time}^{-1}. \]

8. The general case: If “stuff” flowing in \( \mathbb{R}^3 \) with density \( \rho \) and flux density vector \( J \) is manufactured at net rate per unit volume \( f \), then the balance law in integral form is

\[ \frac{d}{dt} \int_B \rho \, dx + \int_{\partial B} J \cdot \nu \, dS = \int_B f \, dx. \] (10)
Moving the time derivative inside and applying the divergence theorem gives us

\[ \int_B \varrho_t \, dx + \int_B \text{div} \, J \, dx = \int_B f \, dx. \]  \hspace{1cm} (11)

If this holds for all closed volumes \( B \), we obtain the differential form of the balance law

\[ \varrho_t + \text{div} \, J = f, \]  \hspace{1cm} (12)

and the conservation law

\[ \varrho_t + \text{div} \, J = 0, \]  \hspace{1cm} (13)

when \( f \equiv 0 \).

9. Example: Let \( \varrho \) be the mass density of a gas that moves by diffusion alone. This means that the gas simply spreads from regions of higher to regions of lower concentration. Since \( -\nabla \varrho \) is the direction in which the density decreases most rapidly, you could reasonably assume that

\[ J = -D \nabla \varrho, \]  \hspace{1cm} (14)

for some positive constant \( D \) with \( [D] = \text{area} \times \text{time}^{-1} \). Plug this into (14) to get

\[ \varrho_t - D \Delta \varrho = 0. \]  \hspace{1cm} (15)

This is the simplest partial differential equation governing diffusion. Since it was derived by Fourier in his study of heat flow, it is usually called the heat equation. Chemists refer to (14) and (15) as Fick’s laws.

10. The constant \( D \) is called the diffusion coefficient. In more sophisticated models, \( D \) is allowed to depend on \( \varrho, x \) or \( t \). For example, if \( D = D(\varrho) \), then (14) becomes

\[ J = -D(\varrho) \nabla \varrho, \]  \hspace{1cm} (16)

and (15),

\[ \varrho_t - \text{div} \, (D(\varrho) \nabla \varrho) = 0. \]  \hspace{1cm} (17)

This is a nonlinear heat equation.

11. Equations (14) and (16) are not laws of nature, but sensible assumptions relating quantities that appear in the mathematical model. Such equations are called constitutive.