A Little Calculus in $\mathbf{R}^n$

1. We denote by

$$x = (x_1, \ldots, x_n),$$

a point or vector in the linear space $\mathbf{R}^n$.

2. **Definition**: The dot product (or inner or scalar product) of vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

3. **Definition**: The (Euclidean) magnitude (or 2-norm) of a vector $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$ is

$$|x| = \left[\sum_{i=1}^{n} x_i^2 \right]^{\frac{1}{2}} = (x \cdot x)^{\frac{1}{2}}.$$

4. **Definition**: We write $f : \mathbf{R}^n \mapsto \mathbf{R}$ if f is a real-valued function whose domain lies in $\mathbf{R}^n$. For example, let $f : \mathbf{R}^4 \mapsto \mathbf{R}$ by

$$f(x_1, x_2, x_3, x_4) = x_1 x_3 - (x_2 + 3x_4)^2.$$

5. **Definition**: Let $f : \mathbf{R}^n \mapsto \mathbf{R}$. The *gradient* of f at x is the vector

$$\nabla f(x) = \left(f_{x_1}(x), \ldots, f_{x_n}(x) \right),$$

where

$$f_{x_i} = \frac{\partial f}{\partial x_i}.$$

6. Let x and h be vectors in $\mathbf{R}^n$. Let $f : \mathbf{R}^n \mapsto \mathbf{R}$. Recall that the derivative of f at x in the direction h is

$$\begin{aligned} D_h f(x) &= \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon} \\ &= \frac{d}{d\varepsilon} f(x + \varepsilon h) \Big|_{\varepsilon = 0} \\ &= \nabla f(x) \cdot h. \end{aligned}$$

7. **Definition**: Let $F_i : \mathbf{R}^n \mapsto \mathbf{R}$ for $i = 1, \ldots, n$. Then

$$F(x) = (F_1(x), , \ldots, F_n(x)),$$

defines a *vector field* on $\mathbf{R}^n$, with *components* $F_1, \ldots, F_n$. In other words, $F : \mathbf{R}^n \mapsto \mathbf{R}^n$. You can think of F as assigning an arrow $F(x)$ to every point x in some domain in $\mathbf{R}^n$. For example, a mass M at the origin $(0,0,0)$ in $\mathbf{R}^3$ creates a gravitational field

$$F(x) = -\frac{GM}{|x|^3}\, x,$$

at $x = (x_1, x_2, x_3)$. So F is a vector field on $\mathbf{R}^3$. It assigns to every point $x \neq 0$ the gravitational field vector pointing back toward the origin.

8. Let

$$F(x) = (F_1(x), \ldots, F_n(x)),$$

be a vector field on $\mathbf{R}^n$. The divergence of the vector field F at x is

$$\operatorname{div} F(x) = \sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_i}.$$

The divergence measures the infinitesimal flux per unit (n-dimensional) volume at x. If $\operatorname{div} F(x)$ is positive, then the net flow of the vector field is *out* of the point x.

9. Note that $\operatorname{div} F : \mathbf{R}^n \mapsto \mathbf{R}$. By the same token, if $f : \mathbf{R}^n \mapsto \mathbf{R}$, then ∇f is a vector field taking $\mathbf{R}^n$ to $\mathbf{R}^n$.

10. **Definition**: Let $F : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a vector field on $\mathbf{R}^n$ and $f : \mathbf{R}^n \mapsto \mathbf{R}$. If

$$-\nabla f(x) = F(x),$$

then f is a *potential* for F. Note that if f is a potential for F, and C is a constant, then $f + C$ is also a potential. A vector field F that has a potential is called *conservative.*

11. Let $\mathbf{R}^n \mapsto \mathbf{R}$ be a function and $F : \mathbf{R}^n \mapsto \mathbf{R}^n$ a vector field with components F_i. Then

$$f(x)F(x) = (f(x)F_1(x), \ldots, f(x)F_n(x)),$$

is also a vector field. Its divergence is and

$$\operatorname{div}(fF) = \nabla f \cdot F + f \operatorname{div} F.$$

12. Let

$$dx = dx_1 \cdots dx_n,$$

be the n-dimensional volume differential. In $\mathbf{R}^2$, $dx = dx_1 d\,x_2$ is the area differential. In $\mathbf{R}^3$, $dx = dx_1\, dx_2\, dx_3$ is the volume element. The integral of $f : \mathbf{R}^n \mapsto \mathbf{R}$ over $B \subseteq \mathbf{R}^n$ is denoted

$$\int_B f(x)\, dx.$$

13. Let B be a blob in $\mathbf{R}^n$. We denote by ∂B the boundary of B. In $\mathbf{R}^2$, ∂B is a curve, and in $\mathbf{R}^3$, a surface. Let dS be the appropriate measure on ∂B. (Thus, dS is the arclength differential in $\mathbf{R}^2$, and the surface area differential in $\mathbf{R}^3$.) Let $\nu = \nu(x)$ be the outer unit normal to B at x and $F : \mathbf{R}^n \mapsto \mathbf{R}^n$ a vector field. The divergence theorem relates the integral over B to an integral over the closed surface ∂B:

$$\int_{\partial B} F \cdot \nu\, dS = \int_B \operatorname{div} F\, dx.$$

The integral on the left-hand side is called the flux of F through the surface ∂B.

14. Integration by parts: Let F be a vector field on $\mathbf{R}^n$ and $f : \mathbf{R}^n \mapsto \mathbf{R}$. Then

$$\begin{aligned}
\int_B F \cdot \nabla f\, dx &= \int_B \left[\operatorname{div}(fF) - f \operatorname{div} F\right]\, dx \\
&= \int_B \operatorname{div}(fF)\, dx - \int_B f \operatorname{div} F\, dx \\
&= \int_{\partial B} fF \cdot \nu\, dS - \int_B f \operatorname{div} F\, dx.
\end{aligned}$$