Dimensional Analysis and the Buckingham Pi Theorem

1. Notation: Denote by a_i and A_j the ith row and jth column of the matrix A. Let $\mathcal{E}$ $\mathcal{L}$, M, $\mathcal{T}$ and $\mathcal{V}$ be the dimensions of energy, length, mass, time and velocity respectively. Denote by $[p]$ the dimensions of a physical quantity p. If for example p is a momentum, then $[p] = \mathcal{M}\mathcal{V}$. This representation is not unique. It is also true that $[p] = \mathcal{M}\mathcal{L}\mathcal{T}^{-1}$ and $[p] = \mathcal{E}\mathcal{V}^{-1}$.

2. The equation $E = mc^2$ is a physical law relating the energy E, the mass m and the speed of light c. You can write it in the form

$$f(E, m, c) = 0, \tag{1}$$

where $f(E, m, c) = E - mc^2$. The dimensions of E, m and c can be expressed in terms of the fundamental dimensions $\mathcal{E}$ and $\mathcal{V}$:

$$[E] = \mathcal{E}, \quad [m] = \mathcal{E}\mathcal{V}^{-2} \quad \text{and} \quad [c] = \mathcal{V}. \tag{2}$$

A quantity Q is *dimensionless* if $[Q] = 1$. For example, $Q = Em^{-1}c^{-2}$ is dimensionless. By (2),

$$[Q] = \left[Em^{-1}c^{-2}\right] = \mathcal{E}\left(\mathcal{E}^{-1}\mathcal{V}^2\right)\mathcal{V}^{-2} = 1.$$

I wrote $[m] = \mathcal{E}\mathcal{V}^{-1}$ instead of $[m] = \mathcal{M}$ in order to avoid introducing a third fundamental dimension. You shoud keep the set of fundamental dimensions as small as possible.

Mathematical modelers have good reason to care about dimensionless quantities. They like to simplify their equations by dropping terms that represent negligible factors. It's hard to know what is negligible when different terms have different dimensions — you're comparing apples and oranges. To get around this problem, you *scale* the equation, that is, makes changes of variable that reduce it to some dimensionless form. With this done, you can compare any terms in the equation and decide which parts are significant and which not.

3. In general, a physical law relating quantities $q_1, \ldots, q_n$ is an equation of the form

$$f(q_1, \ldots, q_n) = 0. \tag{3}$$

The law is *unit-free* if it it doesn't depend on a particular system of units. This will be true of any carefully expressed physical law, so let's not worry about it. The dimensions of the q_j are given as products of powers of fundamental dimensions $L_1, \ldots, L_m$, where $m < n$:

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_m^{a_{mj}}, \tag{4}$$

for $j = 1, \ldots, n$. The exponents a_{ij} are stored in the the dimension matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

***a**. From the first example, let E, m and c be the physical quantities and $\mathcal{E}$ and $\mathcal{V}$ the fundamental dimensions. Write out the dimension matrix.

4. Just as you form the kinetic energy $K = (\text{mass}) \times (\text{velocity})^2$, form a new physical quantity π as the product of powers of the q_j.

$$\pi = q_1^{r_1} \cdots q_n^{r_n}. \tag{5}$$

***a**. Let $x = [r_1 \cdots r_n]^T$. Show that

$$[\pi] = L_1^{a_1 x} \cdots L_m^{a_m x}. \tag{6}$$

* Do for credit.

*__b__. Show that π is dimensionless if and only if $Ax = 0$.

*__c__. Find a nontrivial solution $x \in \mathbf{R}^3$ to $Ax = 0$, where A is the dimension matrix from (3a). Form the corresponding dimensionless quantity π.

5. You now have a recipe for extracting useful dimensionless quantities from a physical law: Use Gaussian (or Gauss-Jordan) elimination to find a nontrivial solution $x \in \mathbf{R}^n$ to the homogeneous system (HS) $Ax = 0$, then form the product (5). Thus to each dimensionless quantity there corresponds a vector x satisfying HE. We call the dimensionless quantities $\pi_1, \ldots, \pi_k$ *independent* if the corresponding vectors $x_1, \ldots, x_k$ are linearly independent. The rank theorem tells you how many independent dimensionless quantities you can extract from the physical law (5). Here's an outline of the argument:

i. Since HS is consistent, the rank theorem is applicable.

ii. The number of linearly independent solutions is equal to the number n_f of free variables. (Think about that one.)

iii. By the rank theorem and (*ii*) there are $n - \text{rank(A)}$ linearly independent solutions to HS.

iv. Each of the $n - \text{rank(A)}$ linearly independent solutions to HS corresponds to a dimensionless quantity.

*__a__. Let A be the dimension matrix for the law $E = mc^2$. Show that there is only one independent dimensionless quantity. (We already know that $Q = Em^{-1}c^{-2}$ is a dimensionless quantity. There are others, e.g. Q^2 and Q^{-1}, but they're not independent of Q.)

6. What do these dimensionless quantities do for you? The Buckingham Pi theorem asserts that if physical quantities $q_1, \ldots, q_n$ and fundamental dimensions $L_1, \ldots, L_m$ (with $m < n$) have dimension matrix A, then you can replace the (unit-free) physical law $f(q_1, \ldots, q_n) = 0$ with an equivalent law

$$F(\pi_1, \ldots, \pi_k) = 0, \tag{7}$$

where the π_j are dimensionsless and $k = n - \text{rank(A)}$.

*__a__. A simplified model of an explosion has an amount e of energy released at a point, causing a spherical blast wave to propagate through air of initial density ρ. The blast wave has radius r at time t. We'll see how to use dimensional analysis to measure e. Assume a unit-free law of the form $f(t, r, \rho, e) = 0$, and give the dimension matrix A, using $\mathcal{T}$, $\mathcal{L}$ and $\mathcal{M}$ as your fundamental dimensions.

*__b__. Use the procedure described in paragraph 5 to determine the number of independent dimensionless quantities. Find those quantities.

*__c__. Use the Buckingham Pi theorem and the result of part (b) to show that

$$r = C\left(\frac{et^2}{\rho}\right)^{1/5}, \tag{8}$$

for some constant C.

d. To find e from (8): You already know ρ. By examining a video of the explosion, determine values r_1 and r_2 of the radius at positive times t_1 and t_2. Plugging each (t, r)-pair into (8) gives two equations in the unknowns C and e. Then solve for e.