Surface Integrals

1. Let $d\sigma$ be the surface area differential on a surface $\mathcal{S}$. If $f : \mathbf{R}^2 \mapsto \mathbf{R}$ is C^1 on a domain R and

$$\mathcal{S} = \{(x, y, z) \mid z = f(x, y) \text{ for } (x, y) \in R\}, \tag{1}$$

then

$$d\sigma = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA. \tag{2}$$

We can thus reduce the integral of a continuous function $g : \mathbf{R}^3 \mapsto \mathbf{R}$ over $\mathcal{S}$ to an integral over R:

$$\int_{\mathcal{S}} g(x, y, z)\, d\sigma = \int_R g(x, y, f(x, y)) \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA, \tag{3}$$

where dA is the area differential on R.

2. We *orient* a surface $\mathcal{S}$ by choosing a unit normal vector $\vec{n}$. (In these notes, we always assume that a surface can be oriented.) If $\mathcal{S}$ is given by (1), the unit normals are

$$\vec{n} = \pm \frac{\langle f_x(x, y), f_y(x, y), -1\rangle}{\sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}}. \tag{4}$$

The one with the plus sign is called *downward pointing*, and the the other, *upward pointing.* We orient $\mathcal{S}$ by choosing one of them to be $\vec{n}$. If $\mathcal{S}$ is a closed surface, we choose either the *outer* or *inner* unit normal.

3. The *flux* of a vector field across an oriented surface $\mathcal{S}$ is

$$\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma. \tag{5}$$

As we saw in class, Φ measures the net flow of $\vec{F}$ through $\mathcal{S}$. Flow "against" $\vec{n}$ is counted as negative, and flow "with" $\vec{n}$ as positive.

4. Suppose that $\mathcal{S}$ is given by (1). Then by (2) and (4),

$$\vec{n}\, d\sigma = \pm \langle f_x(x, y), f_y(x, y), -1\rangle\ dA. \tag{6}$$

Thus,

$$\Phi = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma = \pm \iint_R \vec{F}(x, y, f(x, y))\ \langle f_x(x, y), f_y(x, y), -1\rangle\ dA, \tag{7}$$

where the plus sign indicates the downward orientation, and the minus sign the upward.

5. Formula (7) should be modified in the obvious way when $\mathcal{S}$ is the graph of a function $f(x,z)$, for (x,z) in some region R. In this case,

$$\Phi = \pm \iint_R \vec{F}(x, f(x,z), z) \ \langle f_x(x,y), -1, f_z(x,z)) \rangle \ dA, \tag{8}$$

where dA is the area differential on the xz-plane. The plus and minus signs are for the left and right pointing unit normals respectively. The case $x = f(y,z)$ is handled similarly.

6. Let $\vec{F} = \langle F_1, F_2, F_3 \rangle$ be a C^1 vector field. The *divergence* of $\vec{F}$ is

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = F_{1_x} + F_{2_y} + F_{3_z}. \tag{9}$$

Note that $\operatorname{div} \vec{F} : \mathbf{R}^3 \mapsto \mathbf{R}$. Thus $\operatorname{div} \vec{F}$ is a *scalar valued* function.

7. Let B be a box, centered at P, with volume V. Let the boundary ∂B be oriented so that the unit normal points outward. As we showed in class,

$$\iint_{\partial B} \vec{F} \cdot \vec{n} \, d\sigma = \iiint_B \operatorname{div} \vec{F} \, dV. \tag{10}$$

Divide by the volume V and shrink B to the point P to get

$$\lim_{B \downarrow P} \frac{1}{V} \iint_B \vec{F} \cdot \vec{n} \, d\sigma = \operatorname{div} \vec{F}(P). \tag{11}$$

We may thus interpret the divergence of $\vec{F}$ at P is the "infinitesimal flux" per unit volume of $\vec{F}$ out of P.

8. If $\operatorname{div} \vec{F}(P) > 0$, the point P is called a *source*. If $\operatorname{div} \vec{F}(P) < 0$, P is a *sink*. If $\operatorname{div} \vec{F}(P) = 0$ for all P in a region D, then $\vec{F}$ is called *incompressible* on D.

9. The region B in equation (11) doesn't have to be a box. Any blob that can be shrunk to the point P will do. As it happens, (10) also holds for domains more general than boxes. This is the assertion of the *divergence theorem*.

10. The Divergence Theorem: If $Q \subset \mathbf{R}^3$ is bounded, simply connected and enclosed by ∂Q, $\vec{n}$ is the outer unit normal to ∂Q, and $\vec{F}$ is C^1, then

$$\iint_{\partial Q} \vec{F} \cdot \vec{n} \, d\sigma = \iiint_Q \operatorname{div} \vec{F} \, dV. \tag{12}$$

The idea behind the divergence theorem is simple. Consider an infinitesimal region of volume dV, containing the point (x,y,z). Since the divergence is the infinitesimal flux per unit volume out of a point, the quantity

$$\operatorname{div} \vec{F}(x,y,z) \, dV, \tag{13}$$

is the *net* flow of $\vec{F}$ out of (x, y, z). When we integrate (13), the flow *out* of one interior region *into* another contributes nothing, leaving only the flux out of Q through ∂Q. Hence the conclusion (12).

11. Advice on doing flux integrals: Let $\mathcal{S}$ be an oriented surface with unit nromal $\vec{n}$. Let $\vec{F}$ be a vector field that is C^1 in a simply connected region containing $\mathcal{S}$.

a. If the integral is simple enough, you can use (5). For example, if you have an inverse square field

$$\vec{F}(x, y, z) = \frac{c\vec{r}}{\|\vec{r}\|^3},$$

and $\mathcal{S}$ is the sphere of radius R centered at the origin, then $\vec{n} = \vec{r}/R$ and

$$\begin{aligned}\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma &= \frac{c}{R} \iint_{\mathcal{S}} \frac{\vec{r}}{\|\vec{r}\|^3} \cdot \vec{r}\, d\sigma \\ &= \frac{c}{R^2} \iint_{\mathcal{S}} d\sigma \\ &= 4\pi c.\end{aligned}$$

b. If $\mathcal{S}$ is closed and the direct use of (5) isn't inviting, try the divergence theorem.

c. If $\mathcal{S}$ is not closed, it might be advantageous to replace it with a surface $\mathcal{C}$ that is closed, and then apply the divergence theorem. Suppose for example that you want to compute the flux of

$$\vec{F}(x, y, z) = \langle z - x, x + y, 0 \rangle,$$

across the upper hemisphere $\mathcal{S}$ of radius 1, centered at the origin, oriented upward. Let $\mathcal{D}$ be the disk of radius 1 about the origin in the xy-plane, oriented downward. You can tell at a glance that

$$\iint_{\mathcal{D}} \vec{F} \cdot \vec{n}\, d\sigma = 0. \tag{14}$$

Since $\mathcal{C} = \mathcal{S} \cup \mathcal{D}$ is closed, we can apply the divergence theorem. Let B be the region bounded by $\mathcal{C}$. Then,

$$\begin{aligned}\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma &= \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\mathcal{D}} \vec{F} \cdot \vec{n}\, d\sigma \qquad \text{(by (14))} \\ &= \iint_{\mathcal{C}} \vec{F} \cdot \vec{n}\, d\sigma \\ &= \iiint_{B} \operatorname{div} \vec{F}\, dV \\ &= 0.\end{aligned}$$

d. If necessary, use (7).