Math 208
Surface integrals and the differentials for flux integrals

Our text fails to explicitly state the formulas for $\mathbf{n}\,d\sigma$, instead preferring to give formulas for $\mathbf{n}$ and $d\sigma$ separately. But the proof on page 889 of the formula $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$ on page 890 actually shows that $\mathbf{n}\,d\sigma = \pm\mathbf{r}_u \times \mathbf{r}_v\,du\,dv$ – that is, $\mathbf{r}_u \times \mathbf{r}_v\,du\,dv$ is a vector of length $d\sigma$ and direction normal to the surface, and the only such vectors in 3D are that and its negative! Regarding $\mathbf{n}\,d\sigma$ as the basic formula rather than **n** and $d\sigma$ separately is simpler - when you need $d\sigma$, it can be obtained as simply $|\mathbf{n}\,d\sigma|$, while **n** itself is just the unit vector in the direction of $\mathbf{n}\,d\sigma$. Finding **n** and $d\sigma$ separately and multiplying them when doing a flux integral leads to computing $|\mathbf{r}_u \times \mathbf{r}_v|$ twice, once in the denominator of **n** and once in $d\sigma$, only to cancel these <u>every</u> time! Why do that?

If the surface is given parametrically by $\mathbf{r}(u,v)$, then $\mathbf{n}\,d\sigma = \pm\mathbf{r}_u \times \mathbf{r}_v\,du\,dv$. (1)

Our book also fails to state the formulas for $\mathbf{n}\,d\sigma$ when the surface is written as one variable is a function of the other two. These are the most common cases used, and knowing $\mathbf{n}\,d\sigma$ for those cases saves several steps. Suppose $z = f(x,y)$ with f differentiable. Using y and x as our parameters leads to $\mathbf{r} = < x, y, f(x,y) >$, so
$\mathbf{r}_y \times \mathbf{r}_x = < 0,1, f_y > \times < 1,0, f_x > = < f_x, f_y, -1 >$, giving:

If the surface is part of $z = f(x,y)$, then $\mathbf{n}\,d\sigma = \pm(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dy\,dx$. (2)

Formulas for $\mathbf{n}\,d\sigma$ always have a choice of sign which depends on the orientation of the surface involved. E.g. in (2), + gives downward orientation, − gives upward (since up/down is determined by the **k** coefficient). Also, be aware that the integral can, of course, be done using $dy\,dx$ as indicated, $dx\,dy$, or even $r\,dr\,d\theta$.

Example 1: Find the flux of $\mathbf{F}(x,y,z) = < 3x, 3y, -2z >$ over that portion of the upward oriented paraboloid $x^2 + y^2 - z = 0$ which satisfies $z \le 9$.

Solution: The surface equation gives us $z = x^2 + y^2$, so $f(x,y) = x^2 + y^2$. Since we want upward oriented, $\mathbf{n}\,d\sigma = -(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dy\,dx = < -2x, -2y, 1 >\,dy\,dx$, which leads to:

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,d\sigma = \iint_{R_{xy}} <3x,3y,-2z>\bullet<-2x,-2y,1>\,dy\,dx$$
$$= \iint_{R_{xy}} (-6x^2-6y^2-2z)\,dy\,dx$$
$$= \iint_{R_{xy}} (-8x^2-8y^2)\,dy\,dx$$

But R_{xy} is the region in the *xy*-plane where $z = x^2 + y^2 \le 9$, the inside of the circle of radius 3 centered at the origin, which means this integral is best done in polar coordinates. We get:

$$\iint_{R_{xy}} (-8x^2-8y^2)\,dy\,dx = \iint_{R_{xy}} (-8r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 -8r^3 dr\,d\theta$$
$$= \left(-2r^4\Big|_0^3\right)\left(\theta\Big|_0^{2\pi}\right) = -324\pi \qquad \blacksquare$$

The above example can also be done by parameterizing $\mathbf{r}$ in terms of r and θ. However, as above, it is usually simpler to set up the integral in terms of x and y, and then convert the integral to polar coordinates, rather than find $\mathbf{r}_r \times \mathbf{r}_\theta$.

Note that one can "rotate" the roles of the variables, and get corresponding formulas:

If the surface is part of $x = f(y,z)$, then $\mathbf{n}\,d\sigma = \pm(-\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k})dy\,dz$. (3)

and likewise

If the surface is part of $y = f(x,z)$, then $\mathbf{n}\,d\sigma = \pm(f_x\mathbf{i} - \mathbf{j} + f_z\mathbf{k})dx\,dz$. (4)

Also note that the formulas for $d\sigma$ in these settings (which are implicit in the formulas at the bottom of p. 895 and top of p. 896) are again just the formulas for the lengths of these vector differentials. In general, if you find yourself having trouble memorizing all of the differentials for surface integrals, memorize just the ones for flux (i.e. the $\mathbf{n}\,d\sigma$ formulas), and if you're doing a surface integral which is not a flux integral, find $d\sigma$ by taking the length of the vector part of $\mathbf{n}\,d\sigma$:

Since $\mathbf{n}$ is a unit vector, $|\mathbf{n}| = 1$, so $|\mathbf{n}\,d\sigma| = |\mathbf{n}|\,d\sigma = d\sigma$. Thus:

If the surface is given parametrically by $\mathbf{r}(u,v)$, then $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$. (1a)

If the surface is part of $z = f(x,y)$, then $d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\,dy\,dx$. (2a)

If the surface is part of $x = f(y,z)$, then $d\sigma = \sqrt{1 + f_y^{\,2} + f_z^{\,2}}\; dy\, dz$. (3a)

If the surface is part of $y = f(x,z)$, then $d\sigma = \sqrt{f_x^{\,2} + 1 + f_z^{\,2}}\; dx\, dz$. (4a)

Example 2: Integrate $G(x,y,z) = x\sqrt{y^2+4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$. (This is problem 14 on page 903 of the text.)

Solution: The surface can be written as $z = 4 - \frac{y^2}{4} = f(x,y)$, so

$$\mathbf{n}\, d\sigma = \pm < f_x, f_y, -1 > dy\, dx = \pm < 0, -\tfrac{y}{2}, -1 > dy\, dx$$

Thus $d\sigma = |\mathbf{n}\, d\sigma| = \sqrt{\frac{y^2}{4} + 1}\; dy\, dx = \sqrt{\frac{y^2+4}{4}}\; dy\, dx = \frac{\sqrt{y^2+4}}{2}\, dy\, dx$ and on the surface,

$z = 0 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$. So

$$\iint_S G\, d\sigma = \iint_{R_{xy}} \left(x\sqrt{y^2+4} \right) \frac{\sqrt{y^2+4}}{2}\, dy\, dx = \int_0^1 \int_{-4}^{4} \frac{x(y^2+4)}{2}\, dy\, dx$$

$$= \frac{1}{2}\left(\left.\frac{x^2}{2}\right|_0^1 \right)\left(\left.\frac{y^3}{3} + 4y \right)\right|_{-4}^{4} = \frac{1}{2}\cdot\frac{1}{2}\cdot\left(\frac{64}{3} + 16\right)2 = \frac{56}{3}$$

■

The formulas given in the book for the level surface $g(x,y,z) = c$ are generally harder to use than the formulas above, and the surfaces we use are almost always either easy to parametrize or easy to solve for one of the variables in terms of the other two, so if you know the above formulas and know how to parametrize standard surfaces, you're generally covered. In fact, the formula using $\frac{\pm\nabla g}{|\nabla g \bullet \mathbf{p}|} dA$ for $\mathbf{n}\, d\sigma$ as at the bottom of page 900 is precisely formula (2) above if $\mathbf{p} = \mathbf{k}$ and the partials $f_x = \frac{\partial z}{\partial x}$ and $f_y = \frac{\partial z}{\partial y}$ are calculated implicitly from the level surface equation. Likewise, it is precisely formula (3) if $\mathbf{p} = \mathbf{i}$ or formula (4) if $\mathbf{p} = \mathbf{j}$ and the derivatives are found implicitly.

The only case that is at all common where it is actually advantageous to think in terms of $\mathbf{n}$ and $d\sigma$ separately is in the special case where flux can be found geometrically. If $\mathbf{F}\bullet\mathbf{n} = (\text{scalar component of } \mathbf{F} \text{ in the direction of } \mathbf{n}) = c_1$ is constant on the surface S (note $\mathbf{F}$ can always be written as a vector parallel to $\mathbf{n}$ plus a vector perpendicular to $\mathbf{n}$, and this says the length of the part parallel to $\mathbf{n}$ stays the same on the entire surface) <u>and</u> the surface area of S is known, then $\iint_S \mathbf{F}\bullet\mathbf{n}\, d\sigma = c_1 \cdot (\text{Surface area of } S)$.

Example 3: Find the flux of $\mathbf{F}(x,y,z) = <4x, 4y, 4z>$ across the first octant portion of the sphere $x^2 + y^2 + z^2 = 9$, oriented away from the origin.

Solution: Note that $|\mathbf{F}| = 4|<x,y,z>| = 4\sqrt{x^2+y^2+z^2}$. On this sphere, that means $|\mathbf{F}| = 4\sqrt{9} = 12$. But also, **F** points directly away from the origin, which is the same direction as the outward unit normal **n** to the sphere at that point. Thus at each point on the sphere, $\mathbf{F}\bullet\mathbf{n} = |\mathbf{F}|\cdot|\mathbf{n}|\cos(0°) = |\mathbf{F}| = 12 = c_1$ (since **n** is a unit normal). Also, the surface area of a sphere of radius r is $4\pi r^2$ (easy to remember since it's the derivative of the volume), or in this case 36π, which means the surface area of the first octant portion is $\frac{36\pi}{8} = \frac{9}{2}\pi$. Thus the flux is $12 \cdot \frac{9}{2}\pi = 54\pi$. ■

Flux integral problems

1. Find the flux of $\mathbf{F}(x,y,z) = <2z, 0, 4y^2>$ over the upward oriented portion of $z = 2x + y^2 + 3$ that is defined by $-1 \le x \le 3$ and $1 \le y \le 4$.
2. Find the flux of $\mathbf{F}(x,y,z) = <3, 1, -2>$ over the portion of the surface $y + z^2 = x^2$ that has $-2 \le x \le 2$, $-2 \le z \le 4$ and is oriented toward decreasing *y*.
3. Find the flux of $\mathbf{F}(x,y,z) = <2x, y, z>$ over the finite piece of $x = y^2 z^4$ bounded by $z = -2$, $y = -z$, and $y = 2z$, if the orientation is away from you as viewed from a point on the negative *x*-axis.
4. Find the flux of each of the following through the portion of the cylinder $y^2 + z^2 = 25$ with $-1 \le x \le 2$, if the surface is oriented toward the *x*-axis (i.e., the cylinder is oriented inward):
   a) $\mathbf{F}(x,y,z) = <0, 2y, 2z>$
   b) $\mathbf{F}(x,y,z) = <3xz, -4y, -4z>$

Answers:

1. -240
2. 72
3. -384
4. a) -300π

   b) 600π