1. This part deals with the Jacobian of the fluid flow. Paragraphs 2-4 supply the background material. You should do parts (a)-(f) in paragraph 5.

2. At time \( t \), a fluid particle has position \((x(t), y(t), z(t))\). The initial position is \((\xi, \eta, \zeta)\): \( x(0) = \xi, \ y(0) = \eta \) and \( z(0) = \zeta \).

Think about about spherical coordinates: Given \( \rho, \varphi \) and \( \theta \), you can find corresponding \( x, y \) and \( z \). The same goes for the initial coordinates: Given \( \xi, \eta \) and \( \zeta \) you can find \( x, y \) and \( z \) by following the trajectory of the particle until time \( t \). Thus, you can think of \( \xi, \eta \) and \( \zeta \) as another coordinate system. The Jacobian of the coordinates \( x, y \) and \( z \) with respect to \( \xi, \eta \) and \( \zeta \) (at the time \( t \)) is

\[
J(\xi, \eta, \zeta, t) = \left| \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} \right|.
\]

In class we used the notation

\[
dx \, dy \, dz = \left| \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} \right| \, d\xi \, d\eta \, d\zeta.
\]

It’s a little easier to write

\[
dx \, dy \, dz = |J(\xi, \eta, \zeta, t)| \, d\xi \, d\eta \, d\zeta.
\]

Recall the interpretation: The absolute value of the Jacobian is a local magnification factor. If a set \( S \) of particles initially occupies an infinitesimal region of volume \( d\xi \, d\eta \, d\zeta \), and at time \( t \) an infinitesimal region of volume \( dx \, dy \, dz \), then the ratio of the latter volume to the former is \( |J(\xi, \eta, \zeta, t)| \). For a physical system, \( J(\xi, \eta, \zeta, t) > 0 \), since the particles can’t be compressed into a region of zero or negative volume. (Though it would be very interesting if they could be.) Hence we can drop the absolute values and just write

\[
dx \, dy \, dz = J(\xi, \eta, \zeta, t) \, d\xi \, d\eta \, d\zeta.
\]

3. Let \( \Omega(t) \) be the region at time \( t \) occupied by a set \( S \) of fluid particles. (\( \Omega(t) \) needn’t be a cube.) The volume of the region is

\[
V(t) = \iiint_{\Omega(t)} dx \, dy \, dz.
\]

Bringing in the initial coordinates. The integrand is still 1, the volume element transforms according to (5), and \( \Omega(t) \) is replaced by its image \( \Omega(0) \) in \( \xi\eta\zeta \)-coordinates. Thus

\[
V(t) = \iiint_{\Omega(0)} J(\xi, \eta, \zeta, t) \, d\xi \, d\eta \, d\zeta.
\]

With the \( t \) moved out of the domain of integration, \( V(t) \) is easily differentiated:

\[
\frac{dV}{dt} = \frac{d}{dt} \iiint_{\Omega(0)} J(\xi, \eta, \zeta, t) \, d\xi \, d\eta \, d\zeta = \iiint_{\Omega(0)} \frac{d}{dt} J(\xi, \eta, \zeta, t) \, d\xi \, d\eta \, d\zeta = \iiint_{\Omega(0)} J(\xi, \eta, \zeta, t) \, \frac{\partial}{\partial t} \, d\xi \, d\eta \, d\zeta.
\]

4. You know from part 1 that \( \text{div} \, \vec{v} \) is the density for \( V'(t) \). Therefore,

\[
\frac{dV}{dt} = \iiint_{\Omega(t)} \text{div} \, \vec{v} \, dx \, dy \, dz.
\]
Switch to initial coordinates:

\[ \frac{dV}{dt} = \iiint_{\Omega(0)} J \, \text{div} \, \vec{v} \, d\xi d\eta d\zeta. \]  

(10)

(The divergence \( \text{div} \, \vec{v} \) is a function of \( x, y, z \) and \( t \). But in (10), you have to think of \( x, y \) and \( z \) as functions of \( \xi, \eta \) and \( \zeta \).) By (8) and (10),

\[ \iiint_{\Omega(0)} \dot{J} \, d\xi d\eta d\zeta = \iiint_{\Omega(0)} J \, \text{div} \, \vec{v} \, d\xi d\eta d\zeta. \]  

(11)

Since (11) holds for any initial volume \( \Omega(0) \), the integrands must coincide:

\[ \dot{J} = J \, \text{div} \, \vec{v}. \]  

(12)

(12) is a very useful equation. The Jacobian \( J \) connects the initial state to all subsequent ones, and (12) tells us how \( J \) evolves in time.

5. The derivation of (12) involved a good deal of hand-waving. If the argument left you doubtful, here’s a rigorous derivation.

a. Write out the Jacobian \( J \). You should have a messy combination of products of the partial derivatives of \( x, y \) and \( z \) with respect to \( \xi, \eta \) and \( \zeta \).

b. Differentiate \( J \) with respect to \( t \). You’ll have to apply the product rule to the mess from part (a). This will leave you with an even bigger mess comprising terms like \( x_\eta \) and \( z_\xi \) as well as time derivatives like \( \dot{y}_\zeta \) and \( \dot{x}_\xi \).

c. Since \( \dot{x}, \dot{y} \) and \( \dot{z} \) are the components of the velocity vector of the fluid particle at \( (x, y, z) \) and time \( t \), it must be that

\[ \begin{align*}
\dot{x} &= u(x, y, z, t), \\
\dot{y} &= v(x, y, z, t), \\
\dot{z} &= w(x, y, z, t).
\end{align*} \]  

(13)

This is a system of ordinary differential equations. Use (13) to replace the dotted terms from part (b) with partial derivatives of \( u, v \) and \( w \) with respect to \( \xi, \eta \) and \( \zeta \).

d. Use the chain rule to expand the partial derivatives that appeared in part (c). For example, write out

\[ \frac{\partial}{\partial \xi} v(x, y, z, t), \]  

(14)

where \( x, y \) and \( z \) depend on \( \xi, \eta \) and \( \zeta \). You should now have a truly brobdingnagian mess of an expression for \( \dot{J} \) which contains no \( t \)-derivatives.

e. Sort out the mess from part (d). In particular, you should end up with the lilliputian expression \( J \, \text{div} \, \vec{v} \).

f. Look up brobdingnagian and lilliputian.