The Fourier Transform 5: Eigenfunction Expansions and Transform Pairs

1. There is a connection between transform theory and eigenfunction expansions. Let $f : \mathbf{R} \mapsto \mathbf{C}$ be a smooth function that vanishes identically for $|x| \geq \frac{a}{2}$. The Fourier transform of f is

$$\hat{f}(\xi) = \int f(x) e^{2\pi i \xi x}\, dx.$$

The self-adjoint boundary value problem

$$(P_0) \begin{cases} -iv' = \lambda v, & \text{for } -\frac{a}{2} < x < \frac{a}{2}, \\ v(\frac{a}{2}) - v(-\frac{a}{2}) = 0, & \end{cases}$$

provides us with an orthonormal basis of eigenfunctions

$$e_k(x) = \frac{1}{\sqrt{a}} e^{\frac{2\pi i k x}{a}}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Since f is certainly in $L^2\left[-\frac{a}{2}, \frac{a}{2}\right]$, it has the Fourier exapnsion

$$\begin{aligned} f(x) &= \sum_{k \in \mathbf{Z}} \langle f\, , e_k \rangle e_k(x) \\ &= \sum_{k \in \mathbf{Z}} \left\{ \int_{-\frac{a}{2}}^{\frac{a}{2}} f(z) \bar{e}_k(z)\, dz \right\} e_k(x) \\ &= \sum_{k \in \mathbf{Z}} \left\{ \int_{-\infty}^{\infty} f(z) e^{-\frac{2\pi i k z}{a}}\, dz \right\} \frac{e^{-\frac{2\pi i k x}{a}}}{a} \\ &= \sum_{k \in \mathbf{Z}} \hat{f}\left(\frac{k}{a}\right) \frac{e^{-\frac{2\pi i k x}{a}}}{a}. \end{aligned} \tag{1}$$

For each integer k, let

$$\xi_k = \frac{k}{a},$$

with

$$\Delta \xi = \xi_{k+1} - \xi_k = \frac{1}{a}.$$

With this, (1) becomes

$$f(x) = \sum_{k \in \mathbf{Z}} \hat{f}(\xi_k)\, e^{-2\pi i \xi_k x}\, \Delta \xi. \tag{2}$$

The left-hand side of (2) does not depend on a, and the right-hand side is a Riemann sum for the integral

$$\int \hat{f}(\xi)\, e^{-2\pi i \xi x}\, d\xi. \tag{3}$$

Thus, as $a \to \infty$ (and $\Delta\xi \to 0$) ,we obtain

$$\begin{aligned} f(x) &= \lim_{\Delta\xi \to 0} \sum_{k \in \mathbf{Z}} \hat{f}(\xi_k)\, e^{-2\pi i \xi_k x}\, \Delta\xi \\ &= \int \hat{f}(\xi)\, e^{-2\pi i \xi x}\, d\xi. \end{aligned} \tag{4}$$

This is the conclusion of the Fourier inversion theorem.

2. Example: The self-adjoint problem

$$(P_1) \begin{cases} -v'' = \lambda v, & \text{for } 0 < x < a, \\ v(0) = 0, \\ v(a) = 0, \end{cases}$$

provides the orthonormal basis of eigenfunctions

$$s_k(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi k x}{a}\right), \quad k = 1, 2, 3, \ldots .$$

The Fourier-sine transform of $f \in L^2[0, \infty)$ is

$$f_s(\xi) = \int_0^\infty f(x) \sin(\pi\xi x)\, dx. \tag{5}$$

We can use an argument like the foregoing to derive the inversion formula

$$f(x) = \int_0^\infty f_s(\xi) \sin(\pi\xi x)\, d\xi. \tag{6}$$

3. Example: The self-adjoint problem

$$(P_1) \begin{cases} -v'' = \lambda v, & \text{for } 0 < x < a, \\ v'(0) = 0, \\ v'(a) = 0, \end{cases}$$

provides the orthonormal basis of eigenfunctions

$$s_k(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi k x}{a}\right), \quad k = 0, 1, 2, \ldots .$$

The Fourier-cosine transform of $f \in L^2[0, \infty)$ is

$$f_c(\xi) = \int_0^\infty f(x) \cos(\pi\xi x)\, dx. \tag{7}$$

We can use an argument like the foregoing to derive the inversion formula

$$f(x) = \int_0^\infty f_c(\xi) \cos(\pi\xi x)\, d\xi. \tag{8}$$