Eigenfunction Expansions 2

1. **Notation:** The Kronecker delta is

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$$

2. **Definition:** Let H be a Hilbert space with inner product $\langle u\,, v\rangle$ and norm

$$\|w\| = \sqrt{\langle w\,, w\rangle}.$$

A set $\{v_k\}$ of vectors in H is orthogonal if

$$\langle v_i\,, v_j\rangle = 0 \quad \text{for } i \neq j.$$

3. **Definition:** A set $\{u_k\}$ in H is orthonormal if

$$\langle u_i\,, u_j\rangle = \delta_{ij}.$$

Thus $\{u_k\}$ is orthonormal if it is orthogonal and the u_k are unit vectors.

4. Note that if $\{v_k\}$ is an othogonal set of nonzero vectors, then $\{v_k/\|v_k\|\}$ is orthonormal.

5. **Definition:** The set $\{e_k\}$ in H is an orthonormal basis of H if

a. it is orthonormal and

b. every vector $v \in H$ can be represented as a linear combination of the e_k:

$$v = \sum_k c_k e_k, \tag{1}$$

for scalars c_k.

6. **Example:** The set $\{\vec{i}, \vec{j}, \vec{k}\}$ is a an orthormal basis of $\mathbf{R^3}$. The set $\{\vec{i}, \vec{j}\}$ in $\mathbf{R^3}$ is orthonormal, but is not a basis.

7. If the sum in (1) is infinite, then equality is understood to be in the sense of norm convergence:

$$\left\| v - \sum_{k=1}^{N} c_k e_k \right\| \to 0 \quad \text{as} \quad N \to \infty.$$

8. Let v lie in H, and $\mathcal{O} = \{e_k\}$ be an orthonormal basis of H. Then for scalars c_k,

$$v = \sum_k c_k e_k.$$

By taking inner products, we see that

$$c_m = \langle v\,, e_m \rangle.$$

Thus,

$$v = \sum_k \langle v\,, e_k \rangle e_k. \tag{2}$$

The sum (2) is called the Fourier series of v with respect to $\mathcal{O}$. The numbers

$$\hat{v}_k = \langle v\,, e_k \rangle$$

are the Fourier coefficients.

9. **Proposition:** Let $\mathcal{O} = \{e_k\}$ be an orthonormal basis of H. Then for vectors v and w in H,

$$\langle v\,, w \rangle = \sum_k \langle v\,, e_k \rangle \langle w\,, e_k \rangle. \tag{3}$$

If $v = w$, this becomes

$$\|v\|^2 = \sum_k \langle v\,, e_k \rangle^2. \tag{4}$$

Formulae (3) and (4) are versions of Parseval's identity; (4) is a generalization of the Pythagorean theorem.

10. Let $[a, b]$ be a finite interval. Define the second-oder, linear differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0,$$

where the a_i are smooth and real-valued and $a_2(x) \neq 0$ on $[a, b]$. Let B_1 and B_2 be linear boundary operators of at most the first order. If

$$(P_0) \begin{cases} Lv = \lambda v, & \text{for } a < x < b, \\ B_1 v = 0, & \\ B_2 v = 0, & \end{cases}$$

is self-adjoint, it is called a regular Sturm-Liouville problem. We can use such a problem to generate an orthonormal basis for $L^2[a, b]$. The recipe is

a. Find the eigenvalues $\{\lambda_n\}$.

b. For each eigenvalue, determine the eigenspace E_{λ_n}. (Note that $\dim E_{\lambda_n} \leq 2$.)

c. Find an orthonormal basis $\mathcal{O}_n$ of E_{λ_n}.

d. Since the problem is self-adjoint, the sets $\mathcal{O}_n$ are mutually orthogonal. Hence,

$$\mathcal{O} = \bigcup_n \mathcal{O}_n,$$

is itself an orthonormal set. For a regular Sturm-Liouville problem like (P_0), one can show that $\mathcal{O}$ is actually an orthonormal basis of $L^2[a,b]$. Thus, if

$$\mathcal{O} = \{e_n\},$$

and f is in $L^2[a,b]$ then

$$f = \sum_n \langle f\,, e_n\rangle e_n,$$

where equality is in the sense of L^2.

11. For fixed λ, let $v_1(x,\lambda)$ and $v_2(x,\lambda)$ be linearly independent solutions to the ODE

$$Lv - \lambda v = 0, \tag{5}$$

on (a,b). Then every solution $v(x,\lambda)$ to (5) has the form

$$v(x,\lambda) = c_1 v_1(x,\lambda) + c_2 v_2(x,\lambda), \tag{6}$$

for constants c_1 and c_2. Let

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

and $B(\lambda)$ the 2×2 matrix

$$B(\lambda) = (B_i v_j(\lambda)). \tag{7}$$

We apply the boundary operators to v. Since they are linear,

$$B_1 v(\lambda) = c_1 B_1 v_1(\lambda) + c_2 B_1 v_2(\lambda),$$

and

$$B_2 v(\lambda) = c_1 B_2 v_1(\lambda) + c_2 B_2 v_2(\lambda).$$

Thus,

$$\begin{bmatrix} B_1 v(\lambda) \\ B_2 v(\lambda) \end{bmatrix} = B(\lambda) c. \tag{8}$$

12. **Proposition:** The scalar λ is an eigenvalue if and only if

$$\det B(\lambda) = 0. \tag{9}$$

13. The above proposition gives us a simple algebric procedure for finding the eigenvalues of (P_0). Note that the problem need not be self-adjoint. If the problem is self-adjoint, we can confine our search for eigenvalues to the real line. It is sometimes possible to simplify the search still further. Suppose, for example, that the problem is self-adjoint, and that

$L = -D^2$. Let λ be an eigenvalue with the nontrivial eigenfunction v. Integration by parts shows that

$$\lambda = \frac{\|v'\|_{L^2}}{\|v\|_{L^2}}.$$

Thus the eigenvalues are nonnegative.

14. Example: Consider the self-adjoint problem

$$(P_1)\begin{cases} v'' = \lambda v, & \text{for } 0 < x < 1, \\ v(0) - v(1) = 0, & \\ v'(0) - v'(1) = 0. & \end{cases}$$

By the preceeding paragraph, we know that the eigenvalues are real and nonpositive. We thus set

$$\lambda = -k^2,$$

for $k > 0$. The equation becomes

$$v'' + k^2 v = 0,$$

with general solution

$$v = \begin{cases} c_1 + c_2 x & \text{for } k = 0, \\ c_1 \cos kx + c_2 \sin kx & \text{for } k > 0. \end{cases}$$

It is convenient to think of B as a function of k. When $k = 0$, we take as independent solutions $v_1 = 1$ and $v_2 = x$. Then

$$B(0) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

with

$$\det B(0) = 0.$$

Thus $\lambda_0 = 0$ is an eigenvalue. Any eigenfunction of λ_0 must have the form $v = c_1 + c_2 x$. In order to satisfy the first boundary condition we must set $c_2 = 0$. Hence the eigenspace E_{λ_0} is one-dimensional and has the function

$$c_0(x) \equiv 1,$$

as an orthonormal basis. For $k \neq 0$, we take as independent solutions $v_1 = \cos kx$ and $v_2 = \sin kx$. In this case,

$$B(k) = \begin{bmatrix} 1 - \cos k & -\sin k \\ k \sin k & k(1 - \cos k) \end{bmatrix},$$

with

$$\det B(k) = 2k(1 - \cos k).$$

Thus the nonzero eigenvalues are

$$\lambda_n = -4n^2\pi^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions must be of the form $v = c_1 \cos(2\pi nx) + c_2 \sin(2\pi nx)$. So for $n > 0$, the eigenspace E_{λ_2} is two-dimensional with orthonormal basis $\{c_n(x), s_n(x)\}$, where

$$c_n(x) = \sqrt{2}\cos(2\pi nx),$$

and

$$s_n(x) = \sqrt{2}\sin(2\pi nx).$$

Thus,

$$\mathcal{O} = \{c_0, c_1, s_1, c_2, s_2, \ldots, c_n, s_n, ...\}$$

is an orthonormal basis for $L^2[0,1]$.