Review sheet for Analysis 922: Major Theorems and Definitions

Product Measures and Fubini-Tonelli
Need to include introduction to Product σ-algebras here!

Recall a **Theorem(Fubini-Tonelli):** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Then:

1. **Tonelli:** Let $f \in L^+(X \times Y)$ Then the functions

   $$g(x) = \textstyle\int_Y f_x(y)d\nu(y) \in L^+(X) \quad h(y) = \int_X f^y(x)d\mu(x) \in L^+(Y)$$

   and furthermore

   $$\int_{X\times Y} fd(\mu \times \nu) = \int_X \left(\int_Y fd\nu(y) \right) d\mu(x) = \int_Y \left(\int_X fd\mu(x) \right) d\nu(y). \tag{1}$$

2. **Fubini:** Suppose further that $f \in L^1(\mu \times \nu)$ Then the functions $f_x \in L^1(\nu)$ μ-a.e. $x \in X$ and $f_y \in L^1(\mu)$ ν-a.e. $y \in Y$. Furthermore, the a.e. defined funtions

   $$g(x) = \textstyle\int_Y f_x(y)d\nu(y) \in L^1(X) \quad h(y) = \int_X f^y(x)d\mu(x) \in L^1(Y) \ .$$

   Also, the formula (1) above holds.

Important Remarks and Examples of Fubini-Tonelli:

1. The assumption or σ-finiteness in the above theorem is essential(Recall the example of $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, and $\mu = \mathfrak{m}$, $\nu =$ the counting measure. Consider χ_A where $A = \{(x, x) : x \in [0, 1]\}$).
2. The other assumptions are also essential, please look at homework number 1 for examples.
3. Both Tonelli and Fubini are used together all of the time to switch the order of integration. The following must be verified:
   (a) Verify that $\int |f|d(\mu \times \nu) = \int \int |f|d\mu d\nu = \int \int |f|d\nu d\nu < \infty$ using Tonelli.
   (b) Apply Fubini to $\int \int f \ d\mu d\nu$ to get $\int \int f \ d\nu d\mu$
4. Since even if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete, $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ almost never is(take $A \times E$ where $A \in \mathcal{M}$ and $E \in \mathcal{P}(X) \setminus \mathcal{N}$), we must ask ourselves how Fubini and Tonelli behave when taking the completion of the product measure space. The answer is that all the results that you want to hold are in fact true.

The Lebesgue Measure in $\mathbb{R}^n$: All of the old theorems about Lebesgue sets translate almost verbatim from the one dimensional case to the two dimensional case. We list some of them here:

Theorem: Let $E \in \mathcal{L}^n$. Then we have

1. (a) $\mathfrak{m}(E) = \inf\{\mathfrak{m}(U) | U \subset E,\ U \text{ is open}\}$
   (b) $\mathfrak{m}(E) = \sup\{\mathfrak{m}(K) | K \supset E,\ K \text{ is cpct}\}$
2. $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1 is an F_σ set and A_2 is a G_δ set and N_1, N_2 are both $\mathfrak{m}$-null sets.
3. Let $\mathfrak{m}(E) < \infty$. Then $\forall\ \epsilon > 0$, $\exists$ a finite collection $\{R_j\}_{j=1}^N$ of disjoint rectangles, whose sides are intervals, such that $\mathfrak{m}(E \triangle \bigcup\limits_{j=1}^{N} R_j) < \epsilon$.
4. **Density Theorem for $\mathcal{L}^n$:** If $f \in L^1(\mathfrak{m})$, and $\epsilon > 0$ is given, then there exists a simple function $\phi = \sum\limits_{j=1}^{N} a_j \chi_{E_j}(x)$ where each R_j is a product of intervals, such that $\int_{\mathbb{R}^n} |f - \phi| d\mathfrak{m} < \epsilon$. Moreover, there exists a continuous function g, of compact support, such that $\int_{\mathbb{R}^n} |f - g| d\mathfrak{m} < \epsilon$.
5. $\mathfrak{m}$ is translation invariant. More precisely, if we have $a \in \mathbb{R}^n$ fixed, let $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ be translation by a. Then we have:
   (a) If $E \in \mathcal{L}^n$, $E + a \in \mathcal{L}^n$ and $\mathfrak{m}(E) = \mathfrak{m}(E + a) = \mathfrak{m}(\tau_a(E))$.
   (b) If $f : \mathbb{R}^n \to \mathbb{C}^n$ is $\mathcal{L}^n$-measureable, then so is $f \circ \tau_a$ and furthermore, $\int_{R^n} f \circ \tau_a d\mathfrak{m} = \int_{R^n} f d\mathfrak{m}$.

Here I am leaving out convolutions, and the effect of linear transformations and diffeomorphisms on the Lebesgue integral

Signed Measures

Definition: Let $(X, \mathcal{M})$ be a measureable space. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that:

1. $\nu(\emptyset) = 0$
2. ν assumes only one of $-\infty$ or ∞.
3. If $\{E_j\}$ is a disjoint sequence in $\mathcal{M}$ then $\nu(\bigcup E_j) = \sum \nu(E_j)$ where the series is absolutely convergent if $\nu(\bigcup E_j) \in \mathbb{R}$.

Remarks:

1. Every measure is a signed measure.
2. If μ_1 and μ_2 are measures on $\mathcal{M}$ with at least one of them finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.

3. If μ is a measure and $f : X \to \bar{\mathbb{R}}$ is $\mathcal{M}$-measureable, such that at least one of the integrals $\int f^+ d\mu$ or $\int f^- d\mu$ is finite (such functions are called extended integrable functions), then $\nu(E) = \int_E f d\mu$ is a signed measure.
4. Its a theorem that we proved later that these are the only signed measures.
5. Note that continuity from above and below for measures translates directly to signed measures.

Definition: Let ν be a signed measure on $(X, \mathcal{M})$. Then $E \in \mathcal{M}$ is called a positive (resp. negative or null) set for ν provided $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$ or $\nu(F) = 0$ for all $F \in \mathcal{M}$ with $F \subset E$.

Lemma: Let E be a positive (resp. negative or null) set for ν. Then if $F \in \mathcal{M}$ such that $F \subset E$, F is positive (resp. negative or null) for ν. Furthermore, if $\{E_j\} \subset \mathcal{M}$ is a sequence of positive (resp. negative or null) sets for ν then so is $\bigcup E_j$. (For the proof, the first assertion is clear and the second follows from writing down the union as a disjoint union).

Important Lemma For Hahn Decomposition Theorem: Let $E \in \mathcal{M}$ be such that $0 < \nu(E) < \infty$. Then there exists a positive set $P \subset E$ such that $\nu(P) > 0$. (Note that the proof of this result is very detailed, but it is important to know the Lemma)

The Hahn Decomposition Theorem: Let ν be a signed measure on $(X, \mathcal{M})$. Then $\exists$ a positive set P and a negative set N for ν such that $X = P \cup N$ and $P \cap N = \emptyset$. If P', N' is another such decomposition, then $P \triangle P' = N \triangle N'$ is null for ν. (For the proof, assume wlog that $\nu(X) \neq \infty$, and set $m = \sup\{\nu(E) : E \text{ is positive for } \nu\}$. Then take an increasing sequence of positive sets s/t their union has measure m.)

Definition: The decomposition $X = P \cup N$ into a positive set P and a negative set N is called the Hahn decomposition for ν. It is not unique, for ν-null sets may be transferred from P to N or visa versa.

Definition: We say two signed measures μ and ν on the measureable space $(X, \mathcal{M})$ are **mutually singular**, or μ is **mutually singular** with respect to ν, or visa versa (the defn is symmetric), and we write $\mu \perp \nu$ provided that $\exists$ sets $E, F \in \mathcal{M}$ such that $E = F^c$ and F is null for μ and E is null for ν. Roughly speaking, mutually singular signed measures "live" on disjoint sets whose union is X.

Important Remark: Let $\{P, N\}$ be the Hahm decomposition for ν. Define $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Then it is clear that ν^+ and ν^- are measures and that $\nu(E) = \nu^+(E) - \nu^-(E)$, for every $E \in \mathcal{M}$. Indeed, we have the following theorem:

The Jordan Decomposition Theorem: Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists unique positive measures ν^+ and ν^- on $(X, \mathcal{M})$ such that:

1. $\nu^+ \perp \nu^-$
2. $\nu = \nu^+ - \nu^-$

I.e., if μ^+ and μ^- is another decomposition for μ, then $\nu^+ = \mu^+$ and $\nu^- = \mu^-$. (For the uniqueness part of proof, take the sets from the other decomposition's singular property and use the fact that for Hahn decompositions, we have the symmetric difference of different Hahn decompositions are null sets. Then split up A via both decompositions and measure with μ^+ and μ^-)

Definition: The ν^+, ν^- from the Jordan Decomposition Theorem are called the **positive and negative variations** of ν and $\nu = \nu^+ - \nu^-$ is called the **Jordan Decomposition** for ν. The **total variation** of ν is defined to be the measure $|\nu| = \nu^+ + \nu^-$.

Proposition: Let ν be a signed measure on $(X, \mathcal{M})$. Then

1. E is a ν-null set $\Leftrightarrow$ $|\nu|(E) = 0$.
2. If μ is a signed measure, then $\nu \perp \mu \Leftrightarrow |\nu| \perp \mu \Leftrightarrow \nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof is an easy definition push (it was a HW exercise).

Remarks:

1. If ν omits the value ∞, then $\nu^+(X) = \nu(P) < \infty$, hence ν^+ is a finite measure, and similarly for ν^-.
2. If the range of $\nu \subset \mathbb{R}$, then ν is bounded, namely by $\nu^+(X)$ and $-\nu^-(X)$. In that case, ν has the form $\nu(E) = \int_E f d\mu$ for $E \in M$ where $\mu = |\nu|$ and $f = \chi_P - \chi_N$, where $\{P, N]\}$ is a Hahn Decomposition of ν.

Definition: Let ν be a signed measure on $(X, \mathcal{M})$, $f : X \to \mathbb{C}$ be $\mathcal{M}$-measureable. We say that f is integrable wrt ν, and write $f \in L^1(\nu)$ provided $f \in L^1(\nu^+)$, $f \in L^1(\nu^-)$ and we define $\int f d\nu = \int_P f d\nu^+ - \int_N f d\nu^-$. ν is called finite (resp. σ-finite) provided $|\nu|$ is finite (resp. σ-finite).

Differentiation of measures

Definition: Let ν be a signed measure and μ a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous wrt μ and we write $\nu \ll \mu$, provided $\nu(E) = 0$ for every $E \in \mathcal{M}$ with $\mu(E) = 0$. I.e. $\{$null sets for $\nu\} \subset \{$null sets for $\mu\}$.

Proposition: $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu \Leftrightarrow \nu^+ \ll \mu$ and $\nu^- \ll \mu$ (Homework assignment)

Proposition: In a sense, abs. cty. is the opposite of mutual singularity. Indeed, if we have ν a signed measure and μ a positive measure, then if $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.
Proof: Since $\nu \perp \mu \Leftrightarrow |\nu| \perp \mu$, let P, N be such that $P = N^c$ and $|\nu|(N) = 0$ and $\mu(P) = 0$. Also, note that $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu$. Therefore, we have that $|\nu|(P) = 0$. Therefore $|\nu| = 0$ and hence $\nu = 0$.

Theorem: Let ν be a finite signed measure, and let μ be a positive measure. Then $\nu \ll \mu \Leftrightarrow \forall \epsilon > 0$, $\exists \delta > 0$ such that $|\nu(E)| < \epsilon$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta$.
Proof: Since $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu$ and since $|\nu(E)| \leq |\nu|(E)$, it suffices to prove the result when ν is a positive measure. For the reverse direction, assume that the ϵ, δ condition is satisfied, and let $\epsilon > 0$ be given, and let $E \in \mathcal{M}$ such that $\mu(E) = 0$. Then note that $\mu(E) < \delta$ for every $\delta > 0$, and so $|\nu(E)| < \epsilon$ for every $\epsilon > 0$. Therefore, $\nu(E) = 0$. For the forward direction, assume that the ϵ, δ condition is not satisfied. Then $\exists\ \epsilon > 0$ such that $\forall n \in \mathbb{N}$, there exists E_n with $\mu(E_n) < \frac{1}{2^n}$ and $\nu(E_n) \geq \epsilon$. Set $F_k = \bigcup_{n=k}^{\infty} E_n$, and $F = \bigcap_{k=1}^{\infty} F_k$. Note that the sets F_k are decreasing to F. Now, $\mu(F_k) \leq \sum_{n=k}^{\infty} \mu(E_n) < \sum_{n=k} 2^{-n} = 2^{1-k}$ and so $\mu(F) = 0$ by continuity from above. On the other hand, $\nu(F) = \lim_{k\to\infty} \nu(F_k) \geq \epsilon > 0$ by continuity from above and the fact that ν is a finite measure, so $\nu(F_k) \leq \nu(X) < \infty$ for every $k \in \mathbb{N}$. This says that ν is not abs. cts wrt. μ, a contradiction. Therefore, the ϵ, δ condition is satisfied.

The Radon-Nikodym Theorem: Let ν be a σ-finite signed measure and μ a positive σ-finite measure on $(X, \mathcal{M})$. If $\nu \ll \mu$, then $\exists$ an extended μ-integrable function, $f : X \to \mathbb{R}$ such that $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Furthermore, if g is another such function, then $f = g$ a.e.

The Lebesgue Decomposition Theorem: Let ν be a σ-finite signed measure and μ a positive σ-finite measure on $(X, \mathcal{M})$. Then $\exists$ **unique** σ-finite signed measures ν_0 and ν_1 such that $\nu = \nu_0 + \nu_1$, and $\nu_0 \perp \mu$, $\nu_1 \ll \mu$, and furthermore, there exists an extended μ-integrable function $f : X \to \mathbb{R}$ such that $\nu_1(E) = \int_E f d\mu$ for $E \in M$ and this f is unique a.e.

The theorems above are fundamental, but the proofs are long and complicated so I dont think regurgitating them are very important for a test environment. However, one important consequence of the proof is that if ν is a finite (resp. positive) then both ν_0 and ν_1 are finite (resp. positive). Moreover, in the case that ν is finite, we immediately have that f is $L^1(\mu)$.

Definition: The decomposition $\nu = \nu_0 + \nu_1$, where $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$ where ν is a σ-finite signed measure and μ is a σ-finite positive measure is called the **Lebesgue decomposition** of ν wrt μ.

Remarks: In the case $\nu \ll \mu$ we have shown that $\nu(E) = \int_E f d\mu$ where f is a unique up to definition on null sets extended μ-integrable function by the Radon-Nikodym theorem. This function f is called the **Radon-Nikodym derivative** of ν wrt μ, and we write $\frac{d\nu}{d\mu} = f$. Think of $\frac{d\nu}{d\mu}$ as a class of functions equal to f μ-a.e. Furthermore, it is obvious that $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$ μ-a.e, where ν_1 and ν_2 must omit the same value of infinity. Moreover, one has the chain rule:

Proposition(Chain Rule): Let ν be a σ-finite signed measure, and μ, λ are σ-finite positive measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$. Then we have:

1. If $g \in L^1(\nu)$, then $g \cdot \frac{d\nu}{d\mu} \in L^1(\mu)$, and $\int g d\nu = \int g \cdot \frac{d\nu}{d\mu} d\mu$. (Note that this statement remains valid for $g \in L^+(\nu)$.)
2. Then $\nu \ll \lambda$ and we have the chain rule: $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$.

Corollary: Let μ, λ be positive σ-finite measures on $(X, \mathcal{M})$ such that $\mu \ll \lambda$ and $\lambda \ll \mu$. Then $\frac{d\lambda}{d\mu} = \frac{d\mu}{d\lambda} = 1$ for a.e. λ and μ.

Complex Measures

Definition A $\mathbb{C}$-measure on $(X, \mathcal{M})$ is a function $\nu : X \to \mathbb{C}$ such that $\nu(\emptyset) = 0$ and if $\{E_j\} \subset \mathcal{M}$ is a disjoint sequence in $\mathcal{M}$, then $\nu(\bigcup E_j) = \sum \nu(E_j)$ where the series converges absolutely.

Remarks: Note that in the definition above, infinite values of ν are not allowed. As an example of a complex measure, let $f : X \to \mathbb{C}$ be such that $f \in L^1(\mu)$. Then $\nu(E) = \int_E f d\mu$ is a complex measure. In fact, for any complex measure ν, we write $\nu_r = \Re(\nu)$ and $\nu_i = \Im(\nu)$. Then ν_r and ν_i are both finite signed measures.

Definition: If ν is a $\mathbb{C}$-measure, we define $L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i)$ and if $f \in L^1(\nu)$, we define $\int f d\nu = \int f d\nu_r + i \int f d\nu_i$. If ν, μ are $\mathbb{C}$-measures, we say that $\nu \perp \mu$ provided $\nu_r \perp \mu_r, \nu_r \perp \mu_i, \nu_i \perp \mu_r$, and $\nu_i \perp \mu_i$. We say that $\nu \ll \lambda$ where λ is a positive measure provided $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$. As a remark, there always exists a positive measure λ such that $\nu \ll \lambda$. Indeed, take $\lambda = \nu_r^+ + \nu_r^- + \nu_i^+ + \nu_i^- = |\nu_r| + |\nu_i|$.

The Lebesgue-Radon-Nikodym Theorem for $\mathbb{C}$-measures: Let ν be a $\mathbb{C}$-measure, μ a σ-finite positive measure on $(X, \mathcal{M})$. Then $\exists$ a unique $\mathbb{C}$-measures λ and ρ and $f \in L^1(\mu)$ such that $\lambda \perp \mu$, and $\rho \ll \mu$, $\nu = \lambda + \rho$, where $\rho(E) = \int_E f d\mu$. If g is another such function, then $f = g$ μ-a.e.

Definition: We define the total variation $|\nu|$ for a $\mathbb{C}$-measure ν bu $d|\nu| = |f|d\mu$ when $\nu \ll \mu$ and $d\nu = f d\mu$ where μ is a positive measure. The fact that this construction was well-defined was an exercise.

Properties of $|\nu|$: Let ν be a $\mathbb{C}$-measure on $(X, \mathcal{M})$. Then

1. $|\nu(E)| \leq |\nu|(E), \forall\ E \in \mathcal{M}$.
2. $\nu \ll |\nu|$ and $\left|\frac{d\nu}{d|\nu|}\right| = 1$ $|\nu|$-a.e.
3. $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$ then $\left|\int f d\nu\right| \leq \int |f| d|\nu|$.
4. If ν_1 and ν_2 are $\mathbb{C}$-measures, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Differentiation on $\mathbb{R}^n$

Remember: Things to remember for the measures of balls in $\mathbb{R}^n$: $\mathfrak{m}(B(r,x)) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}$, and $\mathfrak{m}(S(r,x)) = 0$ where $S(r,x)$ denotes the sphere in $\mathbb{R}^n$ of radius r centered at x.

Weiner's Covering Lemma: Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$. Set $U = \bigcup_{B \in \mathcal{C}} B$. If $c < \mathfrak{m}(U)$, then there exists disjoint $B_1, \ldots, B_k \in \mathcal{C}$ such that $\sum\limits_{i}^{k} \mathfrak{m}(B_j) > 3^{-n} c$. This lemma is used to prove several results pertaining to differentiation in $\mathbb{R}^n$.

Definition: A measureable function $f : \mathbb{R}^n \to \mathbb{C}$ is called **locally integrable** with respect to $\mathfrak{m}$ provided $\int_A |f| d\mathfrak{m} < \infty$ for every bounded set $A \subset \mathcal{L}^n$. We set $L^1_{\text{Loc}} = \{f : \mathbb{R}^n \to \mathbb{C}\ :\ f$ is measureable and locally integrable wrt $\mathfrak{m}\}$.

Definition: If f is locally integrable, we define its average value on $B(r,x)$ by:

$$A_r f(x) = \frac{1}{\mathfrak{m}(B(r,x))} \int_{B(r,x)} f d\mathfrak{m}$$

which is well defined for all $x \in \mathbb{R}^n$, $r > 0$.

Lemma: If $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$, then $A_r f(x)$ is continuous on $R^n \times (0, \infty)$, i.e. jointly in x and r.
Proof: Recall that $\mathfrak{m}(S(r,x)) = 0$ for all $x \in \mathbb{R}^n$ and $r > 0$. Fix $x_0 \in R^n$ and $r_0 \in (0, \infty)$, and let $r_k \to r_0$ and $x_k \to x_0$. Then we claim that $\chi_{B(r_k,x_k)}(y) \to \chi_{B(r_0,x_0)}(y)$ pointwise $\forall\ x \in \mathbb{R}^n \setminus S(r_0, x_0)$. Indeed, fix $y \in \mathbb{R}^n \setminus S(r_0, x_0)$, and say that $|y - x_0| < r_0$, the outside the ball case being similar. Set $\epsilon = r_0 - |y - x_0| > 0$. So, there exists $k_0 \in \mathbb{N}$ such that $|r_k - r_0| < \frac{\epsilon}{2}$ and $|x_k - x_0| < \frac{\epsilon}{2}$ for every $k \geq k_0$. Now, for each $k \geq k_0$, we have $|y - x_k| \leq |y - x_0| + |x_0 - x_k| < r_0 - \epsilon + \frac{\epsilon}{2} = r_0 - \frac{\epsilon}{2} < r_k$. Hence, $y \in B(r_k, x_k)$, and so $\chi_{B(r_k,x_k)}(y) = 1 = \chi_{B(r_0,x_0)}(y)$ for k large enough. Therefore, we have shown that $\chi_{B(r_k,x_k)}(y) \to \chi_{B(r_0,x_0)}(y)$ p.w.a.e. Now, choose $k_1 \in \mathbb{N}$ such that $r_k < r_0 + \frac{1}{2}$ and $|x_k - x_0| < \frac{1}{2}$ for all $k \geq k_1$. Then for all $k \geq k_1$ we have that $B(r_k, x_k) \subset B(r_0 + 1, x_0)$, and hence $|\chi_{B(r_k,x_k)}(y)| \leq \chi_{B(r_0+1,x_0)}(y) \in$

$L^1(\mathbb{R}^n, \mathfrak{m})$. Now since $f \in L^1_{\text{Loc}}$, then $\chi_{B(r_k,x_k)}(y)f(y)| \leq \chi_{B(r_0+1,x_0)}(y)|f(y)| \in L^1(\mathbb{R}^n)$, and furthermore $\chi_{B(r_k,x_k)}(y)f(y) \to \chi_{B(r_0,x_0)}(y)f(y)$ p.w.a.e. So, by the Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_{B(r_k,x_k)} f(y)d\mathfrak{m} = \int_{B(r_0,x_0)} f(y)d\mathfrak{m}.$$

This proves the result, since

$$A_{r_k}f(x_k) = \frac{1}{\mathfrak{m}(B(r_k,x_k))}\int_{B(r_k,x_k)} f(y)d\mathfrak{m} \longrightarrow \frac{1}{\mathfrak{m}(B(r_0,x_0))}\int_{B(r_0,x_0)} f(y)d\mathfrak{m} = A_r f(x).$$

Definition: Let $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$. We define its **Hardy-Littlewood maximal function** $Hf(x), x \in \mathbb{R}^n$ by

$$Hf(x) = \sup_{r>0} A_r|f(x)| = \sup_{r>0} \frac{1}{\mathfrak{m}(B(r,x))}\int_{B(r,x)} |f(t)|d\mathfrak{m}(t)$$

By the last lemma, we have that the inverse image of the open rays is open, and so Hf is Borel-measureable, and hence is $\mathcal{L}^n$ measureable (indeed, we have that $(Hf)^{-1}(a,\infty) = \bigcup_{r>0}(A_r|f|)^{-1}(a,\infty)$, which is open in $\mathbb{R}^n$.)

The Hardy-Littlewood Maximal Theorem: There exists $C > 0$ such that $\forall f \in L^1(\mathfrak{m})$, and $\forall \alpha > 0$, we have

$$\mathfrak{m}(\{x \in \mathbb{R}^n \ : \ Hf(x) > \alpha\}) \leq \frac{C}{\alpha}\int_{\mathbb{R}^n} |f|d\mathfrak{m} = \frac{C}{\alpha}||f||_1$$

This proof is fairly reasonable, so should be looked at in detail. It involves the use of the Weiner covering lemma.

Theorem: If $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$, then $\lim_{r\to 0} A_r f(x) = f(x)$ $\mathfrak{m}$-a.e. $x \in \mathbb{R}^n$.
Notes: The proof of this theorem is very involved and uses the Hardy-Littlewood maximal thoerem. Don't expect this on the exam, but its statement and use certainly are important. Note that this theorem implies that we have the following:

$$\lim_{r\to 0^+} \frac{1}{\mathfrak{m}(B(r,x))}\int_{B(r,x)} (f(y) - f(x))d\mathfrak{m}(y) = 0 \quad a.e. x \in R^n$$

We indeed have a stronger statement that we have yet to prove, where we replace $f(y) - f(x)$ with $|f(y) - f(x)|$.

Definition: Let $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$. We define its **Lebesgue Set** to be:

$$L_f = \{x \in \mathbb{R}^n \ : \ \lim_{r\to 0^+} \frac{1}{\mathfrak{m}(B(r,x))}\int_{B(r,x)} |f(y) - f(x)|d\mathfrak{m}(y) = 0\}$$

Theorem: Let $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$. Then $\mathfrak{m}(L_f^c) = 0$.
Proof: Let $c \in \mathbb{C}$. Set $g_c(x) = |f(x) - c|$. So, $g_c(x) \in L^1_{\text{Loc}}$. So, by the previous theorem applied to $g_c(x)$, we have

$$\lim_{r\to 0^+} \frac{1}{\mathfrak{m}(B(r,x))}\int_{B(r,x)} |f(y) - c|d\mathfrak{m}(y) = |f(x) - c| \quad \forall x \in R^n \setminus E_c$$

where E_c is a set of measure zero. Now, since $\mathbb{C}$ is separable (!), let $\{c_j\}$ be a countable dense subset of $\mathbb{C}$. So for each complex number we get a set of measure zero, and we have this countable dense subset lieing around, so define $E = \bigcup_{j=1}^{\infty} E_{c_j}$. Note that $\mathfrak{m}(E) = 0$. Fix $x \notin E$ and let $\epsilon > 0$ be given. Then $\exists\ j \in \mathbb{N}$ such that $|c_j - f(x)| < \epsilon$. Then we have

$$|f(y) - f(x)| \leq |f(y) - c_j| + |f(x) - c_j| < |f(y) - c_j| + \epsilon$$

Hence, we have

$$\limsup_{r \to 0^+} \frac{1}{\mathfrak{m}(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| d\mathfrak{m}(y) \leq \left(\limsup_{r \to 0^+} \frac{1}{\mathfrak{m}(B(r,x))} \int_{B(r,x)} |f(y) - c_j| \right) + \epsilon.$$

This last integral, however, is equal to $|f(x) - c_j| + \epsilon$, as $x \notin E$, so we have the limit being less than 2ϵ for any $\epsilon > 0$, so the limit must be 0. Therefore, $x \in L_f$. Therefore, we have $E^c \subset L_f$, hence $E \supset L_f^c$. Therefore, since $\mathfrak{m}(E) = 0$, we must have that $\mathfrak{m}(L_f^c) = 0$.

Definition: A family $\{E_r\}_{r>0} \subset \mathcal{B}_{R^n}$ is said to **shrink nicely at x** provided

1. $E_r \subset B(r,x) \forall r > 0$
2. $\exists \alpha > 0$ such that $\mathfrak{m}(E_r) > \alpha \mathfrak{m}(B(r,x))$ for all $r > 0$.

The Lebesgue Differentiation Theorem: Let $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$. Then for all $x \in L_f$ we have

$$\lim_{r \to 0} \frac{1}{\mathfrak{m}(E_r)} \int_{E_r} |f(y) - f(x)| d\mathfrak{m}(y) = 0$$

and

$$\lim_{r \to 0} \frac{1}{\mathfrak{m}(E_r)} \int_{E_r} f(y) d\mathfrak{m}(y) = f(x)$$

for every family $\{E_r\}$ that shrinks nicely at x.
Proof: The key point to notice is that we've rigged the definition of "family that shrinks nicely to x" so that the proof of this result is basically already done for us. So, let $x \in L_f$ be fixed. By definition, we have that $E_r \subset B(r,x)$ and $\exists\ \alpha > 0$ such that $\mathfrak{m}(E_r) > \alpha \mathfrak{m}(B(r,x))$. Then immediately, we have

$$0 \leq \frac{1}{\mathfrak{m}(E_r)} \int_{E_r} |f(y) - f(x)| d\mathfrak{m}(y) \leq \frac{1}{\alpha \mathfrak{m}(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| d\mathfrak{m}(y) \to 0$$

as $r \to 0$ since $x \in L_f$. So, the limit exists and is equal to zero for $x \in L_f$. The last statement follows as we may remove the absolute values from the above and move $f(x)$ outside and use definition of $A_r f(x)$.

Add the definition of derivative of a function wrt $\mathfrak{m}$

Definition: A Borel measure ν on $\mathbb{R}^n$ is called **regular** provided

1. $\nu(K) < \infty$ for all $K \subset \mathbb{R}^n$, K compact.
2. $\nu(E) = \inf\{\nu(U) | U \supset E,\ U \text{ open}\}$, for all $E \in \mathcal{B}_{\mathbb{R}^n}$.

A signed or $\mathbb{C}$-Borel measure ν is called **regular** provided $|\nu|$ is regular. As a remark, $n = 1$, 1) $\Rightarrow$ 2) is an old theorem and its true for $n > 1$ as well, but we haven't proved it. Note also that if ν is a regular measure, it already is σ-finite.

Example/Proposition: Let $f \in L^+(\mathbb{R}^n)$ and $\nu(E) = \int_E f d\mathfrak{m}$, $E \in \mathcal{B}_{\mathbb{R}^n}$, then ν is regular $\Leftrightarrow$ $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$.
Proof: Clearly $f \in L^1_{\text{Loc}}(\mathbb{R}^n) \Leftrightarrow$ condition 1) holds above for ν to be regular. We will show 2). Let $E \subset \mathbb{R}^n$ be a bounded Borel set in $\mathbb{R}^n$, and let $\epsilon > 0$ be given. If $G \supset E$ where G is open and bdd, and since $f \in L^1_{\text{Loc}}$, $f \in L^1(G, \mathfrak{m})$. Then by a previous theorem, $\exists \delta > 0$ such that $\int_F f d\mathfrak{m} < \epsilon \ \forall F \subset G$ with F borel and $\mathfrak{m}(F) < \delta$. Recall that $\mathfrak{m}(E) = \inf\{\mathfrak{m}(U) : U \supset E, U \text{ open}\}$. Then there exists an open set V such that $V \supset E$ and $\mathfrak{m}(E) + \delta > \mathfrak{m}(V)$. So, $\mathfrak{m}(U) - \mathfrak{m}(E) < \delta$. Therefore $\mathfrak{m}(U \setminus E) < \delta$. Hence $\int_{U \setminus E} f d\mathfrak{m} < \epsilon$. Or, $\int_U f d\mathfrak{m} < \int_E f d\mathfrak{m} + \epsilon$, therefore $\nu(U) < \nu(E) + \epsilon$. Therefore, 2) above is valid when E is bounded. Now chop up E into bounded pieces for the case where E is unbounded with finite measure (the infinite measure unbounded case is trivial).

Theorem A: Let ν be a regular signed of $\mathbb{C}$-valued Borel measure on $\mathbb{R}^n$, with $d\nu = d\lambda + f d\mathfrak{m}$ where $\lambda \perp \mathfrak{m}$. Then both $d\lambda$ and $f d\mathfrak{m}$ are regular and $d|\nu| = d|\lambda| + |f| d\mathfrak{m}$.

Theorem B: Let λ be a signed or $\mathbb{C}$-borel regular measure on $\mathbb{R}^n$ such that $\lambda \perp \mathfrak{m}$. Then for $\mathfrak{m}$-a.e. $x \in \mathbb{R}^n$ we have $\lim_{r \to 0} \frac{\lambda(E_r)}{\mathfrak{m}(E_r)} = 0$ for every family $\{E_r\}$ that shringks nicely to x. (i.e. $(D\lambda)(x) = 0$ a.e. $x \in \mathbb{R}^n$)

Theorem C: Combining theorems A and B and the Lebesgue Differentiation Theorem we have: Let ν be a regular signed or $\mathbb{C}$-borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f d\mathfrak{m}$ be its Lebesgue-Radon-Nikodym decomposition wrt $\mathfrak{m}$. Then for $\mathfrak{m}$-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{\mathfrak{m}(E_r)} = f(x)$$

for every family $\{E_r\}$ that shrinks nicely to x. Hence $(D\nu)(x) = f(x)$ $\mathfrak{m}$-a.e. $x \in \mathbb{R}^n$.

Applications of the Lebesgue Diff. Theorem to functions of Bounded Variations

Theorem 1: Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Let $G(x) = F(x+) = \lim_{y \to x^+} F(y)$. Then

1. F is cts on $\mathbb{R}$ except at possibly finitely many points, and
2. F and G are differentiable a.e. and $F' = G'$ a.e.

Definition: Let $F : \mathbb{R} \to \mathbb{C}$. Fix an $x \in R$ and let P_x be a partition of the form $P_x = \{x_0 < x_1 < \cdots < x_{n-1} < x_n = x\}$. Denote $\mathcal{P}_x$ to be the collection of all such partitions. For each partition P_x, define $S(P_x, F)$ by:

$$S(P_x, F) = \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$$

Define the **total variation function** of F by:

$$T_F(x) = \sup_{P_x \in \mathcal{P}_x} \{S(P_x, F)\}$$

Remarks:

1. Note that if $P_x \subset \tilde{P_x}$, then $S(P_x, F) \leq S(\tilde{P_x}, F)$. In particular, if $a < b$, let $P_b \in \mathcal{P}_b$ and let $\tilde{P_b} = P_b \cup \{b\}$ (and let $\tilde{\mathcal{P}_b}$ be the collection of all such partitions of this form). Then note that $\mathcal{P}_b \subset \tilde{\mathcal{P}_b}$ since $\tilde{\mathcal{P}_b}$ is just the collection of all partitions of b such that a is included. Therefore, we have $\sup_{P_b \in \mathcal{P}_b}\{S(P_b, F)\} \geq \sup_{\tilde{P_b} \in \tilde{\mathcal{P}_b}}\{S(\tilde{P_b}, F)\}$. However, $S(P_b, F) \leq S(\tilde{P_b}, F)$ for all P_b, so $\sup_{P_b \in \mathcal{P}_b}\{S(P_b, F)\} \leq \sup_{\tilde{P_b} \in \tilde{\mathcal{P}_b}}\{S(\tilde{P_b}, F)\}$, proving that specifying a finite number of points in the partition does not alter the calculation of $T_F(b)$.

2. Also, if $a < b$ then $T_F(b) = \sup_{P \in \mathcal{P}[a,b]} S(P, F) + T_F(a)$. In particular, we have that $T_F(b) - T_F(a) = \sup_{P \in \mathcal{P}[a,b]} S(P, F) \geq 0$. Therefore T_F is a nondecreasing function, with values in $[0, \infty]$.

3. As a matter of notation, we define $T_F(-\infty) = \lim_{x \to -\infty} T_F(x)$ and $T_F(\infty) = \lim_{x \to \infty} T_F(x)$.

Definition: If $T_F(\infty) < \infty$, then F is said to be of **bounded variation** and we denote it by $F \in BV$. So, the space BV is defined by:

$$BV = \{F : \mathbb{R} \to \mathbb{C} : F \text{ is of bounded variation, i.e. } T_F(\infty) < \infty\}$$

Also, we define the total variation of F on $[a, b]$ to be $T_F(b) - T_F(a)$. We denote by $BV[a, b]$ the space of all functions that are of bounded total variation on $[a, b]$. Clearly, if $F \in BV$, we have that $F|_[a, b] \in BV[a, b]$. Conversely, if $F \in BV[a, b]$, we may extend F to an element in BV as follows: $F(x) = F(a) \ \forall \ x < a$, $F(x) = F(b) \ \forall x > b$ (i.e. constant outside $[a, b]$).

Examples:

1. If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in BV$. Indeed, set $x = b$ and $y = a$ such that $a < b$, and let $P \in \mathcal{P}[a, b]$. Then $S(P, F) = \sum |F(x_j) - F(x_{j-1})| = \sum (F(x_j) - F(x_{j-1})) = F(b) - F(a) = F(x) - F(y)$. So, $T_F(x) = \sup_{P_x \in \mathcal{P}_x}\{S(P_x, F)\} = F(x) - F(-\infty) \leq F(\infty) - F(-\infty) < \infty$.

2. Suppose $F, G \in BV$ and $a, b \in \mathbb{C}$. The $aF + bG \in BV$ so that BV is a linear space over $\mathbb{C}$.

3. Suppose $F : \mathbb{R} \to \mathbb{C}$ is differentiable and F' is bounded on $\mathbb{R}$. Then $F \in BV[a, b]$ for any $a < b$. Indeed, let $P \in \mathcal{P}[a, b]$. Then by the mean value theorem, we have that $S(P, F) = \sum |F(x_j) - F(x_{j-1})| = \sum |F'(t_j)|(x_j - x_{j-1}) \leq M \sum (x_j - x_{j-1}) = M(b - a)$. So, $T_F(b) - T_F(a) \leq M(b - a) < \infty$, hence $F \in BV[a, b]$.

4. If $F(x) = \sin(x)$, then $F \in BV[a, b]$ for every $a < b$ but F is not BV.

5. If $F(x) = x \sin(\frac{1}{x})$ for $x \neq 0$ and $F(0) = 0$, then F is not $BV[a, b]$ for any $a < b$ with $0 \in [a, b]$.

Lemma: If $F \in BV$ is $\mathbb{R}$-valued, then $T_F + F$ and $T_F - F$ are nondecreasing on $\mathbb{R}$ (the same statement is valid on $[a, b]$).

Remark: Let $F \in BV$, $\text{Im}(F) \subset \mathbb{R}$. Then $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$, which is the difference of two increasing functions. In fact, we have this as part of the following theorem.

Theorem:

1. $F \in BV \iff \Re(F) \in BV$ and $\Im(F) \in BV$.

2. Let $F : \mathbb{R} \to \mathbb{R}$. Then $F \in BV \;\Leftrightarrow$ F is the difference of two increasing functions on $\mathbb{R}$. Indeed, $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$.
3. If $F \in BV$, then $F(x+)$ and $F(x-)$ exist $\forall x \in R$. In fact so does $F(\pm\infty)$.
4. If $F \in BV$, then the set at which F is discontinuous is at most countable.
5. If $F \in BV$ and $G(x) = F(x+)$, then both F' and G' exist and $F' = G'$ a.e.

Definition: Let $F : \mathbb{R} \to \mathbb{R}$ such that $F \in BV$. The decomposition $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F) := F_1 - F_2$ is called the **Jordan Decomposition** of F, and F_1 and F_2 are called the positive and negative variations of F.

Definition: Let NBV (Normalized Bounded Variation) be the space given by:

$$NBV := \{F \in BV : F \text{ is right continuous and } F(-\infty) = 0\}$$

Let $F \in BV$. Define $G(x) = F(x+) - F(-\infty)$. Then $G(x) \in NBV$. Also, $G' = F'$ a.e.

Lemma: Let $F \in BV$. Then

1. $T_F(-\infty) = 0$
2. If F is also right continuous, then $T_F(x)$ is right continuous as well.

Theorem:

1. Let μ be a $\mathbb{C}$-Borel measure on $\mathbb{R}$. Let $F(x) = \mu((-\infty, x])$. Then $F \in NBV$.
2. Conversely, if $F \in NBV$, then $\exists$ a unique $\mathbb{C}$-Borel measure called μ_F such that $F(x) = \mu_F((-\infty, x])$. Moreover $|\mu_F| = \mu_{T_F}$ (this last statement is left as an exercise).

Proposition: If $F \in NBV$, then $F' \in L^1_{\mathbb{R}}(\mathfrak{m})$. Moreover,

1. $\mu_F \perp \mathfrak{m} \;\Leftrightarrow\; F' = 0$ a.e.
2. $\mu_F \ll \mathfrak{m} \;\Leftrightarrow\; F(x) = \int_{(-\infty,x]} F'(y)d\mathfrak{m}(y)$.

Definition: A function $F : \mathbb{R} \to \mathbb{C}$ is called **absolutely continuous** if $\forall\ \epsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint intervals $\{(a_j, b_j)\}_{j=1}^N$ (need not be open intervals) with $\sum_1^N (b_j - a_j) < \delta$, then $\sum_1^N |F(b_j) - F(a_j)| < \epsilon$. $F : [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$ if the above holds for all choices of $(a, j, b_j) \subset [a, b]$.

Remarks:

1. If $F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous, then F is uniformly continuous. Converse is not true, for example the Contor Function.
2. If $F'(x)$ exists and is bounded on $\mathbb{R}$, then F is absolutely continuous. (choose $\delta = \epsilon/M$ where M is the bound for $|F'|$ and use the mean value theorem.

Proposition: Let $F \in NBV$. Then F is absolutely continuous $\Leftrightarrow \quad \mu_F \ll \mathfrak{m}$. Proof is very long...

Corollary: If $f \in L^1(\mathbb{R}, \mathfrak{m})$, then $F(x) = \int_{(-\infty,x]} f d\mathfrak{m} \in NBV$ and F is absolutely continuous and $F' = f$ a.e. Conversely, if $F \in NBV$ and F is absolutely continuous, then $F' \in L^1(\mathbb{R}, \mathfrak{m})$ and $F = \int_{(-\infty,x]} F' d\mathfrak{m}$.

Lemma: If F is absolutely continuous on $[a,b]$ then $F \in BV[a,b]$.
Proof: Let $\epsilon = 1$. Then $\exists\ \delta > 0$ such that $\forall$ finite disjoint subintervals $\{(a_j, b_j)\}_{j=1}^N$ of $[a,b]$ such that $\sum_1^N (b_j - a_j) < \delta$ then $\sum |F(b_j) - F(a_j)| < 1$. Take $N \in \mathbb{N}$ so that $\frac{b-a}{N} < \delta$. Let $P_0 = \{a = x_0 < x_1 < \cdots < x_N = b\}$ such that $x_j - x_{j-1} = \frac{b-a}{N}$. So, given any partition $P \in \mathcal{P}[a,b]$, say $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$. Set $P_j = (P \cap [x_{j-1}, x_j]) cup \{x_{j-1}, x_j\}$. So, note that $P_j \in \mathcal{P}[x_{j-1}, x_j]$, and so $\sum_{j=1}^N S(P_j, F) \geq S(P, F)$, but $S(P_j, F) = \sum_{j=1}^n |F(\psi_j) - F(\psi_{j-1})| < 1$ by absolute continuity. Hence $S(P, F) < N$, so that $F \in BV[a,b]$.

The Fundamental Theorem of Calculus for Lebesgue Integrals: Let $-\infty < a < b < \infty$ and $F : [a,b] \to \mathbb{C}$ be given. Then the following are equivalent:

1. F is absolutely continuous on $[a,b]$.
2. $F(x) - F(a) = \int_{[a,x]} f d\mathfrak{m}$ for $x \in [a,b]$ for some $f \in L^1([a,b], \mathfrak{m})$.
3. F is differentiable a.e. $[a,b]$, $F' \in L^1([a,b], \mathfrak{m})$ and $F(x) - F(a) = \int_{[a,x]} F' d\mathfrak{m}$, $x \in [a,b]$.

Proof:

- **3)** $\Rightarrow$ **2)**: Trivial
- **2)** $\Rightarrow$ **1)**: Extend f as follows: $\tilde{f}(x) = f(x)$ for $x \in [a,b]$ and $f(x) = 0$ otherwise. Then $\tilde{f} \in L^1(\mathfrak{m})$ on $\mathbb{R}$. Furthermore, $\int_{\mathbb{R}} |\tilde{f}| d\mathfrak{m} = \int_{[a,b]} |f| d\mathfrak{m} < \infty$. Define $\tilde{F}(x) = \int_{(-\infty,x]} \tilde{f} d\mathfrak{m}$. By the previous corollary, $\tilde{F}(x)$ is absolutely continuous on $\mathbb{R}$, and $\tilde{F}'(x) = \tilde{f}$ a.e. on $\mathbb{R}$. But for $x \in [a,b]$, $\tilde{F} = \int_{(-\infty,a]} \tilde{f} d\mathfrak{m} + \int_{[a,x]} \tilde{f} d\mathfrak{m} = \int_{[a,x]} f d\mathfrak{m} = F(x) - F(a)$. So F is absolutely continuous on $[a,b]$, because it is the sum of two absolutely continuous funtions, namely $\tilde{F}(x)$ and $F(a)$, and also note that $F' = \tilde{F}' = f$ a.e. on $[a,b]$.
- **1)** $\Rightarrow$ **3)**: Assume F is absolutely continuous on $[a,b]$. Define $G(x) = F(x)$ for $x \in [a,b]$, and constant from $(-\infty, a]$ and $[b, \infty)$, namely the value of F at a and b respectively. Set $\tilde{G}(x) = G(x) - G(a)$. Then $\tilde{G}$ is absolutely continuous on $\mathbb{R}$, since G was, and therefore $\tilde{G} \in NBV$. By the last corollary, we have that $\tilde{G}'(x) \in L^1(\mathbb{R})$ and $\tilde{G} = \int_{(-\infty,x]} \tilde{G}' d\mathfrak{m}$. However, we know that $\tilde{G}'(x) = G'(x) = F'(x)$ for a.e. $x \in [a,b]$. Hence $F'(x) \in L^1([a,b])$. Moreover, for $x \in (a,b]$, $F(x) - F(a) = \tilde{G}(x) = \int_{(-\infty,x]} \tilde{G}' d\mathfrak{m} = \int_{[a,x]} F' d\mathfrak{m}$, by the definition of $\tilde{G}$ above, which is what we wished to prove.

As a side note, it is important to realize this is yet another characterization of absolute continuity.

Final Remarks:

1. Let μ be a $\mathbb{C}$-Borel measure on $\mathbb{R}^n$. Then μ is called **discrete** if $\exists$ a countable set $\{x_j\} \subset \mathbb{R}^n$ and a sequence $\{c_j\} \subset \mathbb{C}$ such that $\sum |c_j| < \infty$ and $\mu = \sum_1^\infty c_j \delta_{x_j}$ where δ_{x_j} is the point mass at x_j. On the other hand, μ is called **continuous** if $\mu(\{x\}) = 0$ for all singletons $\{x\}$.

2. Any $\mathbb{C}$-Borel measure μ can be written as $\mu = \mu_d + \mu_c$ where μ_d is discrete and μ_c is continuous. To prove this, let $E = \{x \in \mathbb{R}^n | \mu(\{x\}) \neq 0\}$. If F is a countable subset of E then the series $\sum_{x \in F} \mu(\{x\})$ converges absolutely by definition of a $\mathbb{C}$-measure. Let $E_k = \{x \in E \ : \ |\mu(\{x\})| > 1/k\}$. Note that E_k is a finite set for every k, since $\sum_{x \in E_k} \mu(\{x\}) = \mu(E_k)$, and the series converges absolutely. Hence, we have $\infty > \sum_{x \in E_k} |\mu(\{x\})| > \sum_{x \in E_k} = \frac{1}{k}\text{card}(E_k)$, and so $\text{card}(E_k)$ is finite, and so E is at most countable. Now, set $\mu_d(A) = \mu(A \cap E)$, and $\mu_c(A) = \mu(A \cap E^c)$. Clearly $\mu(A) = \mu_d(A) + \mu_c(A)$, for every $A \in \mathcal{B}_{\mathbb{R}^n}$. Furthermore, μ_d is discrete ($\mu_d = \sum \mu(\{x_j\})\delta_{x_j}$ for $\{x_j\} = E$). Finally μ_c is continuous. Indeed, $\mu_c(\{x\}) = \mu(\{x\} \cap E^c) = 0$.

3. If μ is discrete, then $\mu \perp \mathfrak{m}$, and if $\mu \ll \mathfrak{m}$, then μ is continuous. Since $\mu = \mu_d + \mu_c$, and $\mu_c = \mu sc + \mu_{ac}$ where $\mu_{sc} \perp \mathfrak{m}$ and $\mu_{ac} \ll \mathfrak{m}$. Then $\mu = \mu_d + \mu_{sc} + \mu_{ac}$.

4. Notation: Let $F \in NBV$ and μ_F be the unique $\mathbb{C}$-Borel measure inherited by F (i.e.$\mu_F(-\infty, x] = F(x)$, and $\mu_F(x, y] = F(y) - F(x)$. The following notation is useful:

$$\int_{\mathbb{R}} g dF = \int_{\mathbb{R}} g(x) dF(x) = \int_{\mathbb{R}} g d\mu_F$$

Theorem (Integration by Parts Formula for Lebesgue-Stiljes Integrals: Let $F, G \in NBV$, and assume at least one of them is continuous. Then for $-\infty < a < b < \infty$, we have:

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a)$$

Baire Category (in Metric Spaces)

Definition: Let X be a metric space. A set $E \subset X$ is called **nowhere dense** if $\bar{E}$ has empty interior (i.e. $\overset{\circ}{\bar{E}} = \emptyset$). As an example, we have the Cantor Set, since C contains no open interval and C is closed.

Definition: Let $E \subset X$, X a metric space. We say E is **of the first category** if E is a countable union of nowhere dense sets, i.e. $E = \cup_{i=1}^{\infty} E_i$ with the E_i nowhere dense). Otherwise, E is **of the second category**. For some examples, we have that in $\mathbb{R}$, $\mathbb{Q}$ is first category since $\mathbb{Q}$ is countable and single points are nowhere dense in $\mathbb{R}$. Also, nowhere dense sets are first category, so the Cantor set is first category.

Other examples: Let $X = C[0, 1]$ be the set of all $\mathbb{R}$-valued continuous functions on $[0, 1]$, be endowed with the metric $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Recall that $f_n \to f$ in $X \Leftrightarrow f_n \to f$ uniformly in $[0, 1]$.

Definition: Let $f \in C[0,1]$. We define its derivates at $x_0 \in (0,1)$:

$$D^+f(x_0) = \limsup_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{r \to 0^+} \sup_{x_0<x<x_0+r} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D_+f(x_0) = \liminf_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{r \to 0^+} \inf_{x_0<x<x_0+r} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D^-f(x_0) = \limsup_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{r \to 0^-} \sup_{x_0<x<x_0+r} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D_-f(x_0) = \liminf_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{r \to 0^-} \inf_{x_0<x<x_0+r} \frac{f(x)-f(x_0)}{x-x_0}$$

Remark: Clearly, we have that $D^+f(x_0) \geq D_+f(x_0)$ and $D^-f(x_0) \geq D_-f(x_0)$. If all of D^+, D_-, D^-, D_+ are equal and none are $\pm\infty$ we say that f is differentiable at x_0 and we define its derivative at x_0 to be the common value of all the limits above. If $D^+f(x_0) = D_+f(x_0)$ then f is right differentiable and similarly for the left derivates.

Lemma: Let $E = \{\ f \in C[0,1] \mid \exists\ x_0 \in [0, 1-\frac{1}{n}]$ such that $|f(x)-f(x_0)| \leq n(x-x_0)\ \forall x \in [x_0,1)\}$. Then E is first category. The proof of this result is quite involved - perhaps I'll go back and prove part of it later.

Theorem (Baire Category Theorem): Let X be a complete metric space. Then each nonempty open subset of X is of the second category (in particular X is of the second category).

Theorem (Baire's Theorem: Let X be a complete metric space. If one has a sequence $\{\mathcal{O}_k\}_{k=1}^{\infty} \subset \mathcal{P}(X)$ such that each $\mathcal{O}_k$ is open and dense, then $\cap\mathcal{O}_k$ is dense in X.

Remarks/Applications Baire Category Theorem:

1. And complete metric space is second category.
2. Recall one has constructed a continuous function that is nowhere differentiable. Baire Category Theorem gives the existence of such a function. Indeed, since $C[0,1]$ is a complete metric space, then $C[0,1]$ is second category. But the E in the previous example is first category. So we have that $E \subset C[0,1]$ is a proper inclusion. Hence $\exists g \in C[0,1] \setminus E$, so that $D^+g(x_0)$ is infinite for all $x \in [0,1)$, so g is differentiable nowhere.
3. *Proof of Baire Category Theorem assuming Baire's Theorem:* Let U be a nonempty open set in a complete metric space X. Let $\{E_n\}_{n=1}^{\infty}$ be any sequence of nowhere dense sets in X. We shall show that $U \not\subset \bigcup E_n$ as this would imply that U is second category. To show this, note that $\overset{\circ}{\bar{E}} = \emptyset$ for all n. And so, we have that $\{\bar{E_n}^c\}$ is a sequence of open dense subsets of X. Hence by the theorem, we have that $\bigcap(\bar{E_n}^c)$ is dense. So $U \cap \left(\bigcap(\bar{E_n}^c)\right) \neq \emptyset$. Therefore, we have the above desired result by taking complements.
4. The proof of Baire's theorem is pretty technical but should be looked at. Perhaps I'll go back and put it in at a later date.

As an immediate consequence of Baire's Theorems, we have to following:
Theorem: Uniform Boundedness Principle - Special Case: Let X be a complete metric space, and $\mathcal{F} \neq \emptyset$ a collection of real-valued continuous functions on X. Assume that $\forall x \in X$, $\sup\{|f(x)| \mid f \in \mathcal{F}\} = M_x < \infty$. Then there exists an open set $U \subset X$ and a constant $c > 0$ such that $|f(x)| \leq c$ for all $x \in U$ and for all $f \in \mathcal{F}$.

Proof: For $f \in \mathcal{F}$ and $m \in \mathbb{N}$, let $E_{m,f} = \{x \in X \mid |f(x) \leq m\}$. Clearly, $E_{m,f}$ is closed for every m and f. So, if we set $E_m = \bigcap_{f \in \mathcal{F}} E_{m,f}$, E_m is closed. Furthermore, it is clear that $X = \bigcup E_m$ where $m \in \mathbb{N}$. So, since X is complete, not all of the E_m can be nowhere dense as X is second category. So, there exists $m \in \mathbb{N}$ such that $\overset{\circ}{\bar{E}}_m = E_m^\circ \neq \emptyset$. So, there exists an open set $U \subset E_m$ which does the job for us.

Final Remarks:

1. If $E \subset X$ with X a metric space and E first category, then $E^c = X \setminus E$ is called a **residual** set. First category sets are also sometimes called **meager**. Second category sets are called **non-meager** and residual sets are sometimes called **co-meager**.
2. If $\mathcal{O}$ is open and F is closed then $\bar{\mathcal{O}} \setminus \mathcal{O}$ and $F \setminus F^\circ$ are nowhere dense in X. If F is closed and first category, and X is complete, then F is nowhere dense.
3. F is closed and nowhere dense if and only if F contains no nonempty open sets.
4. Let X be a complete metric space. Then $E \subset X$ is residual $\Leftrightarrow$ E contains a dense G_δ set. Hence $E \subset X$ is first category if and only if E is a subset of F where F is an F_σ set such that F^c is dense.

The Basic Theory L^p spaces and a bit on Duality and Separability

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space fixed throughout. For an $\mathcal{M}$ measureable function $f : X \to \mathbb{C}$, and $0 < p < \infty$ we define the p-norm of f to be:

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, \infty]$$

We define the space $L^p(X, \mathcal{M}, \mu)$ or $L^p(X)$ or $L^p(\mu)$ or L^p to be:

$$L^p(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} \mid f \text{ is } \mathcal{M}\text{-measurable and } \|f\|_p < \infty\}$$

Note that as in L^1, $f = g$ in L^p if and only if $f = g$ μ-almost everywhere. Also, if A is a nonempty set and μ is the counting measure on $(A, \mathcal{P}(A))$, we denote l^p to be $L^p(A, \mathcal{P}(A), \mu)$. In the special case where $A = \mathbb{N}$, then $l^p(\mathbb{N})$ will simply be denoted l^p and we have that:

$$l^p = \left\{ \{a_n\} \subset \mathbb{C} \mid \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty \right\}$$

Lemma: The following will be useful for our development of L^p spaces:

1. For $a, b > 0$ and $1 \leq p < \infty$ we have that
$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$
2. For $a, b > 0$ and $0 < p < 1$ then we have
$$(a+b)^p < a^p + b^p$$

Lemma: The above lemma shows that L^p is a vector space for any $0 < p < \infty$, as it gives the bound we need on a sum of elements of L^p; linearity is easily verified.

Remark: The notation $\|f\|_p$ suggests that $\|\cdot\|_p$ is a norm on L^p, i.e. it satisfies the following three properties:

1. $\|f\|_p \geq 0$ for all $f \in L^p(\mu)$ and $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere, which is clearly valid for all $0 < p < \infty$.
2. For every $c \in \mathbb{C}$, we have that $\|cf\|_p = |c| \, \|f\|_p$, which is valid for all $0 < p < \infty$.
3. $\forall \, f, g \in L^p$ we should have $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, the triangle inequality. However this is violated for $0 < p < 1$ (Consider characteristic functions on positive measure sets that are disjoint and use the above lemma).

Lemma(Young's Inequality): The following is key to many results: Let $a, b \geq 0$ and let $0 < \lambda < 1$. Then we have
$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$
with equality if and only if $a = b$.
Important remark: Young's inequality often appears as follows: $a, b \geq 0$, $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. (p, q) are Hölder conjugates). Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with equality if and only if $a^p = b^q$. The following theorem exploits this basic inequality for a powerful theorem:

Theorem (Hölder's Inequality:) Let $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$ be Hölder conjugates. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measureable, then we have
$$\|fg\|_1 = \int_X |f||g| d\mu \leq \|f\|_p \, \|g\|_q = \left(\int_X |f|^p \right)^{1/p} \left(\int_X |g|^q \right)^{1/q}$$
In particular, $f \in L^p$ and $g \in L^q$ implies that $fg \in L^1$, and in this case equality in the above holds if and only if $\alpha |f|^p = \beta |g|^q$, μ-a.e. for some $\alpha, \beta \geq 0$, but both not zero.
Proof: If $\|f\|_p = 0$ or $\|g\|_q = 0$ then $f = 0$ a.e. or $g = 0$ a.e. so $fg = 0$ a.e. so the above is valid. Also, if either of the norms is ∞, again we have that the above is trivial. So, assume that $0 < \|f\|_p < \infty$ and $0 < \|g\|_p < \infty$. Then the above is valid if and only if the following inequality holds:
$$\int \frac{|f||g|}{\|f\|_p \, \|g\|_q} d\mu \leq 1$$

Set $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$. Then by Young's Inequality we have:

$$\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \leq \frac{|f|^p}{p\,\|f\|_p^p} + \frac{|g|^q}{q\,\|g\|_q^q}$$

which is valid for all $x \in X$, with equality if and only if $a^p = b^q$. Integrating over X gives

$$\int_X \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} d\mu \leq \frac{1}{p\,\|f\|_p^p}\int_X |f|^p + \frac{1}{q\,\|g\|_q^q}\int_X |g|^q = \frac{1}{p} + \frac{1}{q} = 1$$

with equality if and only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$, so the result is established by taking $\alpha = \|g\|_q^q$ and $\beta = \|f\|_p^p$.

Note that Hölder's Inequality has immediate consequences: We can now prove the triangle inequality for the p-norms when $1 \leq p < \infty$, which is deemed *Minkowski's Inequality*:

Theorem (Minkowski's Inequality or Triangle Inequality for L^p): Let $1 \leq p \leq \infty$, and $f, g \in L^p$. Then $\|f + g\|_p \leq \|f\|_p + \|g\|_q$.
Proof: The proof is an application of Hölder's inequality twice with $a = |f|$, $b = |f + g|^{p-1}$ and $a' = |g|$, $b' = |f + g|^{p-1}$.

Definition: Let $f : X \to \mathbb{C}$ be $\mathcal{M}$-measureable. We define L^∞ norm to be:

$$\|f\|_\infty = \inf\{a \geq 0 | \mu\{x \in X \mid |f(x)| > a\} = 0\}$$

For such f, set $G_f := \{a \geq 0 | \mu\{x \in X \mid |f(x)| > a\} = 0\}$. Then $\|f\|_\infty = \inf G_f$, with the convention that $\inf \emptyset = \infty$.

Remark: If $G_f \neq \emptyset$, then $\inf G_f = \|f\|_\infty := a_0 \in G_f$. Indeed, by definition we have that $0 \leq a_0 < \infty$. So, there exists a sequence $a_n \in G_f$ such that $a_n \downarrow a_0$. Set $E_n = \{x \in X \mid |f(x)| > a_n\}$. Then $\mu(E_n) = 0$. Also, $E_n \subset E_{n+1}$, as $a_{n+1} < a_n$. And so, note that $\{x \in X \mid |f(x)| > a_0\} = \bigcup E_n$, and so $\mu\{x \in X \mid |f(x)| > a_0\} = 0$, hence $a_0 \in G_f$.

Definition: We define the set of **essentially bounded functions**, L^∞, to be

$$L^\infty = L^\infty(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} | f \text{ is } \mathcal{M}\text{-measureable and } \|f\|_\infty < \infty\}.$$

Note again that we have $f = g$ in L^∞ if and only if $f = g$ μ-almost everywhere.
Remarks:

1. $f \in L^\infty \Leftrightarrow$ there exists a bounded measurable function g such that $f = g$ almost everywhere μ.
2. If $\mu \ll \nu$ and $\nu \ll \mu$ then $L^\infty(\mu) = L^\infty(\nu)$, and the norms also agree.
3. $(L^\infty(\mu), \|\cdot\|_\infty)$ is a normed space. He left the proof of this as an exercise that was not assigned - so it might be a good idea to take a look at it.

Definition: Let $1 \leq p \leq \infty$. We say that $\{f_n\} \subset L^p$ **converges to** f **in** L^p (and we write $f_n \to f$ in L^p) provided $\|f_n - f\|_p \to 0$ as $n \to \infty$. $\{f_n\}$ is **Cauchy** in L^p if $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$.

Theorem: If $f_n \to f$ in L^p $(0 < p < \infty)$, then there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ pointwise almost everywhere.

Proof: Since $\int |f_n - f|^p d\mu \to 0$ as $n \to \infty$, we have that $|f_n - f|^p \to 0$ in L^1. By an old theorem we have a subsequence such that $|f_{n_j} - f| \to 0$ μ-a.e. as $j \to \infty$, as desired.

Definition: Let $(X, \|\cdot\|)$ be a normed space. Let $\{u_n\} \subset X$. We say that $\{u_n\}$ **converges absolutely** (or is **absolutely convergent**) in X if $\sum \|u_n\|$ is convergent.

Theorem: A normed space $(X, \|\cdot\|)$ is complete $\Leftrightarrow$ Every absolutely convergent series in X converges in X. The proof is a little involved and really lives in the realm of functional analysis. Maybe I'll go back in and type it out later.

Theorem (Reisz-Fisher): L^p is a Banach space (i.e. a complete normed space) for $1 \le p \le \infty$.
Proof: Omitted for now.

A Density Theorem for certain simple functions in L^p: Let $1 \le p \le \infty$. Then the following set:

$$\mathcal{S} := \{f = \sum_{j=1}^{n} a_j \chi_{E_j} \mid \mu(E_j) < \infty\}$$

is dense in L^p. To prove this, use the old theorem on approximation of a measureable function by simple functions and note that if we are approximating a function in L^p, we must have that these simple functions are in $\mathcal{S}$.

Theorem (Facts about L^∞:)

1. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measureable, then $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$ then $\|fg\|_1 = \|f\|_1 \|g\|_\infty \Leftrightarrow |g(x)| = \|g\|_\infty$ a.e. on the set where $f \neq 0$.
2. $\|\cdot\|_\infty$ is a norm on L^∞.
3. $\|f_n - f\|_\infty \to 0 \;\Leftrightarrow\; \exists$ a set E such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E. Note that this is strictly stronger than almost uniformly.
4. L^∞ is a Banach space.
5. The simple functions are dense in L^∞.

Proof: The proof of this result was also left as an exercise.

Theorem: If $0 < p < q < r \le \infty$, then we have $L^q \subset L^p + L^r$ (i.e. each $f \in L^q$ may be written as the sum of an L^p function and an L^r function).
Proof: Let $f \in L^q$. Set $E = \{x \in X : |f(x)|| > 1\}$, and let $g = f\chi_E$ and $h = f\chi_{E^c}$. Chearly we have $f = g + h$. Note also that we have chosen g and h appropriately so that the inequalities work and $g \in L^p$ and $h \in L^r$.

Theorem: If $0 < p < q < r <= \infty$ then $L^p \cap L^r \subset L^q$, and futhermore, we have the following inequality:

$$\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$$

where λ is given by $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$ (actually, $\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}$). The proof is a touchy Hölder's inequality application with $\alpha = \frac{p}{\lambda q}$ and $\beta = \frac{r}{(1-\lambda)q}$. I'll go back and type this up later.

Proposition: Let A be any nonempty set and μ be the counting measure on A. If $0 < p \leq q \leq \infty$ then $l^p(A) \subset l^q(A)$ and $\|f\|_q \leq \|f\|_p$.
Proof: In this case, $\mu(E) = 0 \Leftrightarrow E = \emptyset$. So, let $f \in l^p(A)$. Then we have that

$$\|f\|_\infty = \left(\sup_{\alpha \in A} |f(\alpha)|\right)^p = \sup_{\alpha \in A} |f(\alpha)|^p \leq \sum_{\alpha \in A} |f(\alpha)|^p = \|f\|_p^p < \infty.$$

Hence we have that $l^p(A) \subset l^\infty(A)$. So, suppose that $q < \infty$. Then by the last theorem, with $0 < p < q < r = \infty$, we have that $\|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda}$ and $\lambda = \frac{p}{q}$. Therefore, we have that $\|f\|_q \leq \|f\|_p^{p/q} \|f\|_\infty^{1-p/q} \leq \|f\|_p^{p/q} \|f\|_p^{1-p/q} = \|f\|_p$ as desired.

Proposition: If $\mu(X) < \infty$ and $0 < p < q \leq \infty$ then $L^q(\mu) \subset L^p(\mu)$ and $\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}$.
Proof: The $q = \infty$ case is almost trivial and for $q < \infty$ you use Hölder's Inequality again thinking of $|f|^p$ as $1 \cdot |f|^p$ with $\alpha = \frac{q}{p}$ and $\beta = \frac{q}{q-p}$.

Final Remarks:

1. The most important L^p spaces are L^1, L^2 and L^∞. L^2 enjoys many fantastic properties seen later. L^1 and L^∞ are related, but studying practical problems is more fruitful to do in L^p where p is not 1 or ∞.

2. A Banach space B is a normed space such taht the inherited metric is complete.

Duality in L^p spaces

Definition: Let X, Y be Banach spaces over a field $\mathbb{F}$, where $\mathbb{F}$ is either the real or complex numbers. A linear mapping is called a **linear operator** from X to Y. We say a linear operator $T : X \to Y$ is **bounded** if $\|T\| := \sup_{x \in X}\{\|Tx\| \mid \|x\| \leq 1\} < \infty$. The set of all bounded linear operators from X to Y is denoted $\mathcal{L}(X, Y)$, with the understanding that $\mathcal{L}(X) = \mathcal{L}(X, X)$.
Remarks:

1. If $T \in \mathcal{L}(X, Y)$, then it can be shown that the following equalities hold:

$$\begin{aligned} \|T\| &= \sup_{x \in X}\{\|Tx\| \mid \|x\| \leq 1\} \\ &= \sup_{\|x\|=1} \|Tx\| \\ &= \sup_{\|x\| \neq 0} \left\{\frac{\|Tx\|}{\|x\|}\right\} \\ &= \inf\{c \geq 0 \mid \|Tx\| \leq c\|x\| \ \forall x \in X\} \end{aligned}$$

   Here, $\|T\|$ is called the norm of the operator T.

2. If $T \in \mathcal{L}(X, Y)$ then $\|Tx\| \leq \|T\|\|x\|$ for all $x \in X$, as is seen by the second equality applied to $\frac{x}{\|x\|}$, which has norm 1.

3. If $T \in \mathcal{L}(X,Y)$ then $T(0) = 0$.

Definition: A linear mapping T from X into $\mathbb{F}$ is called a **linear functional**.

Proposition: Let X, Y be normed spaces and $T : X \to Y$ be a linear operator. Then TFAE:

1. T is continuous at 0.
2. T is continuous at x for every $x \in X$.
3. T is bounded.

Proof: Will fill in later.

Remark: The special case when $T \in \mathcal{L}(X,\mathbb{F})$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, i.e. T is a bounded linear functional on X, we have the definition of the norm of T is slightly simplified:

$$\|T\| = \sup_{x \neq 0} \frac{|Tx|}{\|x\|} = \sup_{|x|=1} |Tx| = \sup_{|x| \leq 1} |Tx|$$

$$= \inf\{c \geq 0 \mid |Tx| \leq c\,\|x\|\, \forall x \in X\}$$

Theorem (Duality of L^p spaces): Let $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q(\mu)$ is fixed. Then $\phi_g : L^p \to \mathbb{C}$ defined $\phi_g(f) = \int fg d\mu$ satisfies:

1. $\phi_g \in \mathcal{L}(L^p, \mathbb{C})$
2. $\|\phi_g\| = \|g\|_q$.

In addition, if μ is semifinite, the same conclusions hold when $q = \infty$. In fact, we have a much stronger result:

Theorem: Let $1 \leq p, q \leq \infty$ be Hölder conjugates. Let $g : X \to \mathbb{C}$ be a fixed $\mathcal{M}$-measureable function such that $fg \in L^1$ for all $f \in \Lambda$ where $\Lambda = \{$simple functions that are identically zero on E^c with $\mu(E) < \infty\}$, and assume that $M_g := \sup\{\left|\int fg d\mu\right| : f \in \Lambda, \|f\|_p = 1\} < \infty$. Further, assume that either $S_g = \{x \mid g(x) \neq 0\}$ is σ-finite or else assume μ is semifinite. Then $g \in L^q(\mu)$ and $M_g = \|g\|_q$.
Proof: The proof of this result is highly technical.

Reisz Representation Theorem: Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\phi \in L^p(\mu, \mathbb{C})$. Then

1. If $1 < p < \infty$ then $\exists!\ g \in L^q(\mu)$ such that $\phi(f) = \int fg d\mu$ and $\|\phi\| = \|g\|_q$.
2. The same conclusions hold if $p = 1$ provided μ is σ-finite.

Proof: The proof of this result probably will not need to be reproduced.

Definition: The space of all bounded linear functionals on X, $\mathcal{L}(X,\mathbb{F})$ is called the dual space of X, denoted X^*. So, in the case when $X = L^p$, we denote the space of all bounded linear functionals on L^p taking values in $\mathbb{C}$ to be $L_p(\mu)^*$.

Important Remarks:

1. For $1 \leq p < \infty$ (and if $p = 1$ we also assume σ-finiteness of (X, μ), the mapping $F : L^q(\mu) \to L^p(\mu)^*$ sending an L^q function to its functional is an isometric isomorphism. This means that it is a linear bijective continuous map that preserves norms and whose inverse is also continuous. Recall that continuous is the same as bounded in this context.

2. The assertion that μ is σ-finite in the above theorem in the case that $p = 1$ and $q = \infty$ is essential.

3. The mapping $F : L^1(\mu) \to L^\infty(\mu)^*$ where $g \mapsto \phi_g$ is always an isometric injection, but rarely a surjection.

   Proof: First, note that $C[0,1] \subset L^\infty([0,1])$. Let us define $\phi : C[0,1] \to \mathbb{C}$ by $f \mapsto f(0)$. Then ϕ is clearly linear and the norm is 1. But you can use dominated convergence theorem to get a contradiction if you assume that there is an L^1 function such that $\phi(f) = \int f g d\mu$.

Definition: Let X be a normed space over $\mathbb{C}$ or $\mathbb{R}$ and X^* its dual. Then $(X^*, \|\cdot\|)$ is a normed space, and in fact it is always a Banach space(!). We say a sequence $\{x_n\} \subset X$ converges weakly to $x \in X$ if for all $\phi \in X^*$ we have $\phi(x_n) \to \phi(x)$.

Theorem (Uniform Boundedness Principal, General Case): Let X, Y be normed spaces. Let $\{T_a\}_{a \in A} \subset \mathcal{L}(X, Y)$ be a family of bounded linear operators from X to Y. Then

1. If $\sup_{a \in A} \|T_a(x)\| := M_x < \infty$ for all $x \in E$ where E is some second category set in X then $\sup \|T_a\| = M < \infty$.

2. If X is a Banach space, and if $\sup_{a \in A} \|T_a(x)\| := M_x < \infty$, then $\sup \|T_a\| = M < \infty$. Note that if we prove 1, part 2 follows immediately since X Banach implies that X is second category.

Proof: For $n \in \mathbb{N}$, put $E_n = \{x \in X \mid \sup_{a \in A} \|T_a(x)\| \leq n\} = \bigcap_{a \in A} \{x \in X \mid \|T_a(x)\| \leq n\}$. Then $E_n \subset E_{n+1}$ and E_n is closed for all n (by continuity of T_a). Also, we have that $\bigcup_{n=1}^\infty E_n \supset E$ so since E is second category, $\exists n_0 \in \mathbb{N}$ such that $\bar{E}_{n_0}$ is second category. By an exercise, E_{n_0} contains a nonempty open set, say $B(r, x_0)$. So, $E_{n_0} \supset \bar{B}(r, x_0)$ (note my puny attempt at a bar above the ball). I claim that $E_{2n_0} \supset \bar{B}(r, 0)$. Indeed, suppose $\|x\| \leq r$. Then note that $\|x + x_0 - x_0\| = \|x\| \leq r$. Hence we have that $x + x_0 \in \bar{B}(r, x_0) \subset E_{n_0}$. Hence, we have that $\|T_a(x + x_0)\| \leq n_0$, for all $a \in A$. So, we have $\|T_a(x)\| = \|T_a(x + x_0) - T_a(x_0)\| \leq \|T_a(x + x_0)\| + \|T_a(x_0)\| \leq 2n_0$. Therefore $x \in E_{2n_0}$. So, we have shown that for all $x \in X$ with $\|x\| \leq r$ we have $\|T_a(x)\| \leq 2n_0$, for all $a \in A$. So, let $x \in X$ such that $\|x\| \leq 1$. Then $\|rx\| \leq r$. Thus, by the above, we have $\|T_a(x)\| = \left\|\frac{1}{r} T_a(rx)\right\| \leq \frac{2n_0}{r}$ for all $a \in A$. Hence, $\|T_a\| = \sup_{\|x\| \leq 1} \|T_a(x)\| \leq \frac{2n_0}{r}$ for all $a \in A$. Hence we have that $\sup_{a \in A} \|T_a\| < \infty$, as desired.

Definition: Let X be a normed space and X^* its dual. Then $(X^*, \|\cdot\|)$ is a normed space. In fact, $(X^*, \|\cdot\|)$ is a Banach space. We may consider X^{**}, the dual of X^*, where $X^{**} = \mathcal{L}(X^*, \mathbb{C}) = \mathcal{L}(\mathcal{L}(X, \mathbb{C}), \mathbb{C})$. We call X^{**} the **double dual** of X.
For each $x \in X$ we assign an element $\psi_x \in X^{**}$ such that $\psi_x : X^* \to \mathbb{C}$ is given by $\psi_x(\phi) \mapsto \phi(x)$. This is often called the evaluation map. Then we have that $\|\psi_x\| = \sup_{\|\phi\|=1} |\psi_x(\phi)| = \sup_{\|\phi\|=1} \phi(x) \leq \|phi\| \|x\| = \|x\|$. In fact, it can be shown that $\|\psi_x\| = \|x\|$. So the mapping ψ is linear and an isometry. Therefore, ψ is an isometric isometry from X to $\psi(X)$, which is a linear subspace of X^{**}. We say that X is **reflexive** if $\psi(X) = X^{**}$.

Corollary to Reisz Representation Theorem: If $1 < p < \infty$, then $L^p(\mu)$ is reflexive.

Brief mention of Separability of L^p

Theorem (Stone-Weierstrauss): Let $\Omega = \bar{\mathcal{O}}$, where $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^n$, and let $\mathcal{A} \subset C(\Omega)$, where $C(\Omega)$ is the set of continuous functions on Ω. Then $\mathcal{A}$ is dense in $C(\Omega)$ provided $\mathcal{A}$ satisfies:

1. $\mathcal{A}$ is an algebra (i.e. $f, g \in \mathcal{A}$ implies $f+g, fg$ and $cf \in \mathcal{A}$).
2. If $f \in \mathcal{A}$ then $\bar{f} \in \mathcal{A}$.
3. If $x, y \in \Omega$, $x \neq y$ then $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.
4. If $x \in \Omega$ then $\exists f \in \mathcal{A}$ such that $f(x) \neq 0$.

Corollary: Let Ω be as above. Let $\mathcal{P}_{\text{rat}} := \{$ all polynomials on $\mathbb{R}^n$ with rational complex coefficients$\}$. Then $\mathcal{P}_{\text{rat}}$ is dense in $C(\Omega)$.

Corollary: $C(\Omega)$ where Ω is as above is separable.

Theorem: $L^p(\mathbb{R}^n, \mathfrak{m})$ is separable, if $1 \leq p < \infty$.
Proof: Let $B_j = B(j, 0) := \{x \in \mathbb{R}^n \mid |x| \leq j\}$. Set $\mathcal{P}_{\text{rat}}$ to be as above. Set $\mathcal{P}_j = \{\chi_{B_j} Q \mid Q \in \mathcal{P}_{\text{rat}}\}$. Then $\mathcal{P}_j$ is dense in $C(B_j)$, and $\bigcup \mathcal{P}_j$ is countable. Let $f \in L^p(\mathbb{R}^n, \mathfrak{m})$, and let $\epsilon > 0$ be given. Then $\exists \phi \in C_0(\mathbb{R}^n)$ such that $\|f - \phi\| < \frac{\epsilon}{3}$, since the continuous functions of compact support are dense in L^p. Say that $\phi =$ outside B_k for some $k \in \mathbb{N}$. Now approximate ϕ with a polynomial Q in the above countable union of countable sets and playing with the triangle inequality gives you that $\|f - Q\| < \epsilon$, as desired.