Review sheet for Analysis 921: Major Theorems and Definitions

Measure Theory

Definition: Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **measure** on $\mathcal{A}$ if the following conditions hold:

1. $\mu(\emptyset) = 0$
2. μ is countably additive on $\mathcal{A}$, i.e., $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$, where $E_i \in \mathcal{A}$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$.

Definition: μ is said to be **finite** if $\mu(X) < \infty$.

Definition: If $X = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < \infty$ then μ is said to be σ**-finite**. If a subset E of X has this property, then E is called a set of σ-finite measure.

Definition: If $\forall E \in \mathcal{A}$ with $\mu(E) = \infty$, $\exists F \in \mathcal{A}$ such that $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called **semifinite**. Note that σ-finite implies semifinite by a homework problem (look at a σ-finite decomposition of the set E in question).

Properties of Measures

1. **Monotonicity:** $E \subset F$ implies that $\mu(E) \leq \mu(F)$
2. **Subadditivity:** $\{E_j\} \subset \mathcal{A}$ implies that $\mu(\bigcup_{j}^{\infty} E_j) \leq \sum_{j}^{\infty} \mu(E_j)$.
3. **Continuity from Below:** If $\{E_j\} \subset \mathcal{A}$ increases to E, then $\mu(E) = \lim_{j} \mu(E_j)$.
4. **Continuity from Above:** If $\{E_j\} \subset \mathcal{A}$ decreases to E and $\mu(E_1) < \infty$, then $\mu(E) = \lim_{j} \mu(E_j)$.

Outer Measures

Definition: A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is said to be an **outer measure** if the following conditions are satisfied:

1. $\mu^*(\emptyset) = 0$
2. $A \subset B$ implies that $\mu^*(A) \leq \mu^*(B)$ (Monotonicity)
3. If $\{A_j\} \subset \mathcal{P}(X)$, then $\mu^*(\bigcup_{j} A_j) \leq \sum_{j} \mu^*(A_j)$ (Subadditivity)

Theorem 1: Let $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset \in \mathcal{E}, X \in \mathcal{E}$ and let $\rho : \mathcal{E} \to [0, \infty]$ be such that $\rho(\emptyset) = 0$. Then $\forall A \in \mathcal{P}(X)$, define

$$\mu^*(A) = \inf\{\sum_j \rho(E_j) : E_j \in \mathcal{E} \text{ such that } A \subset \bigcup_j E_j\}$$

Then μ^* is an outer measure on X.

Definition: Suppose that $X \neq \emptyset$ and let μ^* be an outer measure on X. A set $A \subset X$ is called **μ^*-measureable** (or just measureable) if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subset X$$

Note that since $E = (E \cap A) \cup (E \cap A^c)$, subadditivity of μ^* gives $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, so that one half of the inequality is satisfied. The other inequality is also trivially satisfied when $\mu^*(E) = \infty$. So, we also have the following equivalent definition for μ^*-measureable set:

$$A \text{ is } \mu^*\text{-measureable } \Leftrightarrow \forall E \subset X \text{ with } \mu^*(E) < \infty, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem 2 (Carathéodory's Theorem): Let $X \neq \emptyset$, and μ^* be an outer measure on X, and let $\mathcal{M} = \{A \subset X : A \text{ is } \mu^*\text{-measureable}\}$. Then $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Definition: Let $\mathcal{S} \subset \mathcal{P}(X)$ be an algebra (remember thats a collection like a σ-algebra except closed under finite unions and complements). A function $\mu_0 : \mathcal{S} \to [0, \infty]$ is a **premeasure** on $\mathcal{S}$ provided:

1. $\mu_0(\emptyset) = 0$
2. Whenever $\{A_j\} \subset \mathcal{S}$ is such that $\bigcup_j A_j \in \mathcal{S}$, $\mu_0(\bigcup_j A_j) = \sum_j \mu_0(A_j)$.

Note that premeasures are automatically monotone, finitely additive, and they induce an outer measure on $\mathcal{P}(X)$:

$$\mu^*(A) = \inf\{\sum_j \mu_0(E_j) : E_j \in \mathcal{S} \text{ such that } A \subset \bigcup_j E_j\}$$

Indeed, we have the following theorem:

Theorem 3: Let μ_0 be a premeasure on an algebra $\mathcal{S}$, and μ^* be the induced outer measure mentioned above. Then we have:

1. $\mu^*|_{\mathcal{S}} = \mu_0$.
2. $\mathcal{S} \subset \mathcal{M}$, i.e., every set in $\mathcal{S}$ is μ^*-measureable.

Theorem 4: Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{S}$, and let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{S}$. Let μ^* be the outer measure induced by μ_0. Then there exists a measure μ on $\mathcal{A}$ such that $\mu = \mu^*|_{\mathcal{A}}$ and $\mu|_{\mathcal{S}} = \mu_0$. Moreover, if ν is another measure on $\mathcal{A}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{A}$ with equality if $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on A.

Measure Theory on $\mathbb{R}$ and $\mathbb{C}$:

For our picture, we will take $\mathcal{E} = \{\text{all h-intervals of } \mathbb{R}\}$, where an h-interval looks like $(a, b]$ where $-\infty \leq a < b < \infty$, or (a, ∞) where $-\infty \leq a < \infty$ or $\emptyset$.

Note that $\mathcal{E}$ is an elementary family. Let $\mathcal{S}$ be the algebra generated by all finite disjoint unions of elements in $\mathcal{E}$. Let $\mathcal{A}(= \mathcal{B}_{\mathbb{R}}) = \mathcal{M}(\mathcal{S})$ (where $\mathcal{M}(\mathcal{S})$ is the smallest σ-algebra containing $\mathcal{S}$. So, we have $\mathcal{E} \subset \mathcal{S} \subset \mathcal{A} = \mathcal{B}_{\mathbb{R}}$. We need only to invent a premeasure on $\mathcal{S}$ to get the ball rolling. So,

Theorem: Let $F : \mathbb{R} \to R$ be a monotone increasing and right continuous function ($F(x+) = F(x)$), and let $\{I_j\} = \{(a_j, b_j]\}_j^n$ be a disjoint union of h-intervals. Define $\mu_0(\bigcup_j I_j) = \sum_j^n F(b_j) - F(a_j)$, $\mu_0((-\infty, b]) = F(b) - \lim_{s\to\infty} F(s)$, or $\mu_0((a, \infty)) = \lim_{s\to\infty} F(s) - F(a)$. Then μ_0 is a premeasure on $\mathcal{S}$.

Note that the proof is highly technical and difficult to reproduce without notes, so probably won't be tested on. The statement of the theorem and how to use it, however, are important.

So, there exists and outer measure μ^* on $\mathbb{R}$ given by $\mu^*(E) = \inf\{\sum_j \mu_0(A_j) : \{A_j\}_j \text{ is an } \mathcal{S} \text{ cover of } E\}$. In fact, more explicitly, we have:

$$\mu^*(E) = \inf\{\sum_j F(b_j) - F(a_j) : \{(a, j, b_j]\}_j \text{ covers } E\}$$

Furthermore, Carathéodory implies that $\mu = \mu^*|\mathcal{M}$ is a complete measure and also Theorem 4 implies that $\mu|_{\mathcal{B}_{\mathbb{R}}}$ is a measure.

Remarks: Measures of some Trivial Sets: Below we list some measures of various trivial sets. The proof technique is the same for each fact, use either continuity from above or below of the measure to obtain the desired fact.

$$\mu_F(\{a\}) = F(a) - F(a-) \qquad \mu_F([a, b]) = F(b) = F(a-)$$

$$\mu_F(([a, b)) = F(b-) - F(a-) \qquad \mu_F((a, b)) = F(b-) - F(a)$$

Lemma 1: Let $E \in \mathcal{M}$. Recall that $\mu^*(E) = \inf\{\sum_j F(b_j) - F(a_j) : \{(a, j, b_j]\}_j \text{ covers } E\}$. It is in fact true that we may remove the half open intervals and replace them with open intervals:

$$\mu^*(E) = \inf\{\sum_j F(b_j) - F(a_j) : \{(a, j, b_j)\}_j \text{ covers } E\}$$

Lemma 2(Very Useful): For $E \in \mathcal{M}$, the following two equalities hold:

$$\mu(E) = \inf\{\mu(U) \ : \ U \text{ open}, U \supset E\}$$

$$\mu(E) = \inf\{\mu(K) \ : \ K \text{ compact}, K \subset E\}$$

Theorem: Let $E \subset \mathbb{R}$. Then TFAE:

1. $E \in \mathcal{M}$
2. $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.
3. $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

Theorem: (Invariance of translation and well behaved stretching of the Lebesgue measure) Let $E \in \mathcal{L}$. Recall that $(\mathcal{L}, \mathfrak{m})$ is the complete σ-algebra that we get from Carathéodory's theorem and the discussion above when $F(x) = x$. Then $E + s, rE \in \mathcal{L}$ and furthermore, we have $\mathfrak{m}(E + s) = \mathfrak{m}(E)$ and $\mathfrak{m}(rE) = |r|\mathfrak{m}(E)$.

Construction of a Lebesgue Non-measurable set: Define $\sim$ on $[0, 1)$ by $x \sim y \iff x - y \in \mathbb{Q}$. Set $\mathcal{S} = \{[x] : x \in [0, 1)\}$ to be the set of equivalence classes of $[0, 1)$ under $\sim$. By the Axoim of Choice, there exists $f : \mathcal{S} \to \bigcup [x]$ such that $f([x]) \in [x]$. Set $N = \text{im}(f)$. Let $\{r_j\}$ be an enumeration of $\mathbb{Q} \cap [0, 1)$, and define

$$x \oplus y = \begin{cases} x + y & x + y < 1 \\ x + y - 1 & x + y \geq 1 \end{cases}$$

Set $E \oplus y = \{x \oplus y : x \in E\}$. Define sets $N_j = N \oplus r_j$. Then $N_m \cap N_n = \emptyset$ and $\bigcup_j N_j = [0, 1)$. Also, E $\mathfrak{m}$-measureable implies tat $E \oplus y$ is $\mathfrak{m}$-measureable. Now look at what $\bigcup_j N_j = [0, 1)$ implies when $\mathfrak{m}$ is applied to both sides. Also, note that the above proof may be altered to show that **any** set of positive measure contains a set that is non-measureable.

Measureable Functions

Definition: Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $f : X \to Y$ any function. f is said to be $(\mathcal{M}, \mathcal{N})$-measureable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Note that compositions of measureable functions are measureable if the σ-algebras agree and if X and Y are metric spaces, then continuous functions $f : X \to Y$ are $(\mathcal{B}_X, \mathcal{B}_Y)$-measureable.

Theorem: Let $c \in \mathbb{R}$, $f, g : X \to \mathbb{R}$ are $\mathcal{M}$-measureable functions, and $\psi : \mathbb{R} \to \mathbb{R}$ a continuous function. Then $cf, f + g, f \circ \psi$, and fg are all $\mathcal{M}$-measureable.

Theorem: Let $f_j : X \to \bar{\mathbb{R}}$ be a sequence of $\mathcal{M}$-measureable functions. Then the following functions:

$$g_1(x) = \sup_j f_j(x) \qquad g_2(x) = \inf_j f_j(x)$$
$$g_3(x) = \limsup_j f_j(x) \quad g_4(x) = \liminf_j f_j(x)$$

are $\mathcal{M}$-measureable. Hence, if $f(x) = \lim_j f_j(x)$, then f is $\mathcal{M}$-measureable.

Definition: We define a function $\chi_E(x)$, called the **characteristic function** of the set E to be:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Note that $\chi_E(x)$ is $\mathcal{M}$-measureable $\Leftrightarrow E \in \mathcal{M}$.

Definition: A **simple function** f on X is a finite $\mathbb{R}$-linear combination of characteristic functions on sets in $\mathcal{M}$: $f(x) = \sum_j r_j \chi_{E_j}(x)$. We often simplify our life by assuming that $\bigcup_j E_j = X$ and $\{E_j\}$ are disjoint, by glueing sets and characteristic functions together if the same real coefficient is used, and including the complement of the sets E_j with a zero coefficient, if not all of X is represented.

Approximation of Measureable Functions by Simple Functions Theorem: Let $(X, \mathcal{M})$ be a measureable space and $f : X \to \bar{\mathbb{R}}$ be $\mathcal{M}$-measureable. Then there exists a sequence $\{\phi_n\}$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |\phi_n| \leq \cdots \leq |f|$ with $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set where f is bounded.

Lemma: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Then the following hold:

1. If f is $\mathcal{M}$-measureable and $f = g$ almost everywhere then g is $\mathcal{M}$-measureable.
2. If $\{f_n\}$ is $\mathcal{M}$-measureable for all n, and $f_n \to f$ a.e., then f is $\mathcal{M}$-measureable.

Lemma: Let $(X, \mathcal{M}, \mu)$ be a complete measure space and $(X, \bar{\mathcal{M}}, \bar{\mu})$ its completion. If f is $\bar{\mathcal{M}}$-measureable, then there exists a function g that is $\mathcal{M}$-measureable and $g = f$ $\bar{\mu}$-a.e.

Lemma: If $f : X \to \bar{\mathbb{R}}$ is a function and $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\bar{\mathbb{R}}}$, then f is measureable iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Lemma: If $f : (X, \mathcal{M}) \to (Y, \mathcal{B}_Y)$ is a function with Y is a separable metric space, i.e., the topology induced by the metric has a countable basis, then f is $(\mathcal{M}, \mathcal{B}_Y)$-measureable iff $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the countable basis for the topology on Y.

Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ with $f = f_1 + if_2 := \Re(f) + i\Im(f)$. Then

1. If f is measureable, then f_1, f_2, and $|f|$ are measureable.
2. f_1 and f_2 measureable implies that f is measureable.

Integration of Functions:

Definition(Integration of Simple Functions): For this definition, first we define the following sets:

$$L^+ := \{f : X \to [0, \infty] \mid f \text{ is } \mathcal{M}\text{-measureable}\}$$

$$S^+ := \{\phi : X \to [0, \infty] \mid \phi \text{ is simple}\}$$

$$S_f^+ := \{\phi : X \to [0, \infty] \mid \phi \text{ is simple and } 0 \leq \phi \leq f\}.$$

Now let $\phi \in S^+$, with standard representation $\phi = \sum_j^n a_j \chi_{E_j}(x)$, we define

$$\int \phi \, d\mu = \sum_j^n a_j \mu(E_j)$$

with the convention that $0 \cdot \infty = 0$ kept in mind. Also, if $\phi \in S^+$, then $\chi_A \phi \in S^+$, so we define $\int_A \phi \; d\mu = \int \phi \chi_A \; d\mu = \sum_j^n a_j \; \mu(A \cap E_j)$.

Proposition: Let $\phi, \psi \in S^+$. Then the following are valid:

1. If $c \geq 0$, then $\int c\phi = c \int \phi$.
2. $\int (\phi + \psi) \; d\mu = \int \phi \; d\mu + \int \psi \; d\mu$.
3. If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.
4. If one defines $\nu(A) = \int_A \phi \; d\mu$ for $A\mu$-measureable, then μ is a measure on $\mathcal{M}$.

Definition(Integration of Nonnegative Functions:) Let $f \in L^+$. Then we define

$$\int f \; d\mu = \sup\{\int \phi \; d\mu : \phi \in S_f^+\}$$

Note that pretty immediate from the definitions, we get the following two facts:

1. If $c \geq 0$, then $\int cf \; d\mu = c \int f \; d\mu$.
2. If $f, g \in L^+$ and $f \leq g$ then $\int f \leq \int g$ (as its a sup over a bigger set).

The Monotone Convergence Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\} \subset L^+$ such that $f_n \leq f_{n+1}$ and $f = \lim_n f_n$. Then we have the corresponding convergence of integrals:

$$\lim_n \int f_n \; d\mu = \int f \; d\mu (= \int \lim_n f_n d\mu)$$

Theorem: $\int$ is both finitely and countably additive on L^+. I.e, Let $\{f_n\}$ be a finite or infinite sequence in L^+. Then we have

$$\int \sum_j f_j = \sum_j \int f_j$$

Lemma: If $f \in L^+$, then $\int f \; d\mu = 0 \;\Leftrightarrow\; f = 0$ almost everywhere.

Corollary: The MCT remains valid if the convergence is pointwise almost everywhere, instead of everywhere. I.e., if $(X, \mathcal{M}, \mu)$ is a measure space and $\{f_n\} \subset L^+$ such that $f_n \leq f_{n+1}$ and $f = \lim_n f_n$ almost everywhere. Then we have the corresponding convergence of integrals:

$$\lim_n \int f_n \; d\mu = \int f \; d\mu (= \int \lim_n f_n d\mu)$$

Fatou's Lemma: Let $\{f_n\}$ be any sequence in L^+. Then

$$\int [\liminf_n f_n(x)] d\mu \leq \liminf_n \int f_n d\mu$$

Fatou's Lemma Again: Let $\{f_n\} \subset L^+$, $f \in L^+$ with $f_n \to f$ a.e., then $\int f \leq \liminf_n \int f_n$.

Theorem: If $f \in L^+$ with $\int f d\mu < \infty$, then f is finite a.e., and $\{x|f(x) > 0\}$ is σ-finite.

Definition(Integration of Complex Valued Functions): For $f : X \to \bar{\mathbb{R}}$ a measureable function, if $\int f^+ d\mu$ **OR** $\int f_- d\mu$ is finite, we define $\int f d\mu = \int f^+ d\mu - \int f_- d\mu$. If both integrals are finite, we say f is **integrable**. Note that f is integrable if and only if $|f|$ is measureable and $\int |f| d\mu < \infty$.

Notation: Set $L^1_r = \{f : X \to \bar{\mathbb{R}} \mid f \text{ is measureable and } \int |f| < \infty\}$. We have proved in class that L^1_r is a real vector space and $\int$ is a linear functional on it. Furthermore, set $L^1 = \{f : X \to \mathbb{C} | f \text{ is measureable and } \int |f| < \infty\}$. We have proved in class that L^1 is a complex space and $\int$ is a linear functional on it. Note that f is integrable if and only if $\Re(f)$ and $\Im(f)$ are integrable.

Proposition: If $f \in L^1$, then $|\int f| \leq \int |f|$.

Sketch of Proof: Set $\alpha =$ complex conjugate of $\text{sgn}(\int f)$. I.e. $\alpha = \frac{\bar{\int f}}{|\int f|}$. Then note that $|\int f| = \alpha \int f \in \mathbb{R}$, so we have:

$$|\int f| = \alpha \int f = \Re(\alpha \int f) = \Re(\int \alpha f) = \int \Re(\alpha f) \leq \int |\alpha f| = |\alpha| \int |f| = \int |f|$$

as desired.

Theorem: Suppose $f, g \in L^1(X, \mu)$. Then the following conclusions are valid:

1. $\{x|f(x) \neq 0\}$ is a μ-σ-finite set.
2. $\int_E f = \int_E g \ \forall E \in \mathcal{M} \iff \int |f - g| = 0 \iff f = g$ μ-a.e.

Important Remarks: It follows from the last theorem that if f is a measureable function that is defined on a set $E \in X$ such that $\mu(E^c) = 0$, then we can still define the integral $\int f$ to be $\int g$ where g is an extension of f to all of X in any manner we choose. With this in mind, it is more convenient to think of L^1 as the set of all equivalence classes where the equivalence relation is equality a.e. This "new" L^1 is still a complex vector space and $\int$ is a linear functional on it. This "new" L^1 gives a 1-1 correspondence between $L^1(\mu)$ and $L^1(\bar{\mu})$, by an old theorem.

Definition: Let $f, g \in L^1(\mu)$. Define a **metric** on L^1 by:

$$\rho(f, g) = \int |f - g| d\mu$$

The fact that it is a metric is clear from the linearity of $\int$. Hence, we may think of L^1 as a (later complete) metric space, and if $\{f_n\}$ is a sequence in L^1 then we say that $\{f_n\} \to f$ **in** L^1 if $\rho(f_n, f) \to 0$ as $n \to \infty$, i.e., $\int |f_n - f| d\mu \ \to 0$ as $n \to \infty$.

The Dominated Convergence Theorem: Let $\{f_n\} \subset L^1$ with $f_n \to f$ p.w.a.e, and suppose that there exists a $g \in L^1 \cap L^+$ such that $|f_n| \leq g$ a.e. for all n. Then $f \in L^1$ and

$$\int f = \int [\lim_n f_n] d\mu = \lim_n \int f_n d\mu$$

i.e., we may interchange the "limiting operations" of lim and $\int$.

Theorem: Let $\{f_j\} \subset L^1$ such that $\sum\limits_n^\infty \int |f_j| d\mu < \infty$. Then we immediately we have $\sum\limits_n^\infty f_n$ converges almost everywhere to some function $f \in L^1$. Moreover, we have:

$$\int f d\mu = \int \sum_n^\infty f_n d\mu = \sum_n^\infty \int f_j d\mu$$

again, allowing us to interchange the "limiting operations".

Density Theorem 1: The set of simple integrable functions is dense in L^1. More precisely, $\forall f \in L^1$, there is a sequence $\{\phi_n\} \subset S^1$(simple integrable functions) such that $\phi_n \to f$ in L^1. Hence, we have that $\lim\limits_n \int |\phi_n - f| \to 0$.

Density Theorem 2: Let $f \in L^1(\mathbb{R}, \mu)$ where μ is a Lebesgue-Stiljes measure, and let $\epsilon > 0$ be given. Then by the previous theorem, there exists a simple integreble function $\phi = \sum\limits_{j=1}^n a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. Then the sets E_j can be taken to be a finite union of disjoint open intervals by an old theorem, and moreover, there exists a continuous function g vanishing outside a bounded (open) interval such that $\int |f - g| < \epsilon$. Hence, the continuous functions of compact support are also dense in L^1.

Theorem (How to relate Riemann's Integral with Lebesgue's): Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then we have:

1. If $f \in \mathcal{R}[a, b]$, then f is Lebesgue measureable on $[a, b]$ and of course $f \in L^1([a, b], \mathfrak{m})$, as f is bounded. Moreover, we have $\int_{[a,b]} f d\mathfrak{m} = \int_a^b f dx$.
2. $f \in \mathcal{R}[a, b] \;\Leftrightarrow\; \mathfrak{m}(\{x | f \text{ is discontinuous at } x\}) = 0$.

Important Remark on Improper Riemann Integrals: (Note this argument may be altered for any improper Riemann integral, not just the infinite cases) Assume $f \in \mathcal{R}[0, b]$, for all $b > 0$, and assume $f \in L^1[0, \infty)$. Then we have

$$\int_{[0,\infty)} f d\mathfrak{m} = \int_0^\infty f dx = \lim_b \int_0^b f dx$$

This result may be easily proved with the Dominated Convergence Theorem. However, the assumption that $f \in L^1$ is essential, for the limit on the RHS of the above equation may exist while the Lebesgue integral may not.

Modes of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space. So far, we are familiar with a convergent sequence of functions $\{f_n\}$ in the following senses:

1. $\{f_n\} \to f$ pointwise a.e.
2. $\{f_n\} \to f$ uniformly
3. $\{f_n\} \to f$ in L^1. I.e. $\lim\limits_n \int |f_n - f| d\mu \;\to 0$.

Examples: We will begin by investigating several examples that show that the different modes of convergence don't necessarily imply the others, except under special conditions. For example, set $f_n = \frac{1}{n}\chi_{[0,n)}$, $g_n = \chi_{(n,n+1)}$, and $h_n = n\chi_{[0,1/n]}$, which converge to 0 uniformly, p.w., and p.w.a.e. respectively. However, for each n, $\int |f_n| = \int |g_n| = \int |h_n| = 1$, so that f_n, g_n and h_n do not converge to 0 in L^1.

Example/Theorem: If $f_n \to f$ a.e. and $|f_n| \leq g \in L^1$ for all n, then $f_n \to f$ in L^1.
Proof: First, note that $|f| = \lim |f_n| \leq |g|$ a.e. Also, note that $f_n \to f$ a.e. implies that $|f_n - f| \to 0$ a.e. Hence, $|f_n - f| \leq 2|f| \leq 2|g| \in L^1$, hence DCT implies that $\int |f_n - f| \to 0$ as $n \to \infty$, as desired.

Definition: Let $f_n : X \to \mathbb{C}$ be a sequence of $\mathcal{M}$-measureable functions. We say that $f_n \to f$ **in measure** if $\forall \epsilon > 0$, $\mu(\{x \in X \ : \ |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$.
Alternative Definition via Homework: $f_n \to f$ in measure iff $\forall \epsilon > 0$, there exists an N such that $n \geq N$ implies $\mu(\{x \in X \ : \ |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$

Definition: Let $f_n : X \to \mathbb{C}$ be a sequence of $\mathcal{M}$-measureable functions. We say that f_n is **Cauchy in measure** if $\forall \epsilon > 0$, $\mu(\{x \in X \ : \ |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0$ as $m, n \to \infty$.
Alternative Definition via Homework: f_n is Cauchy in measure iff $\forall \epsilon > 0$, there exists an N such that $m, n \geq N$ implies $\mu(\{x \in X \ : \ |f_n(x) - f_m(x)| \geq \epsilon\}) < \epsilon$

Note that in the above examples f_n, g_n converge to 0 in measure, but h_n is not Cauchy in measure.

Theorem 1: If $\{f_n\} \to f$ in L^1, then $\{f_n\} \to f$ in measure.
Proof: Let $\epsilon > 0$ be given. Then set $E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Then we have $\mu(E_{n,\epsilon}) = \int_{E_{n,\epsilon}} 1 d\mu \leq \frac{1}{\epsilon} \int_{E_{n,\epsilon}} |f_n(x) - f(x)| d\mu \leq \frac{1}{\epsilon} \int |f_n(x) - f(x)| d\mu \to 0$ as $n \to \infty$, since $f_n \to f$ in L^1.

Theorem 2: Let $\{f_n\}$ be Cauchy in measure. Then there exists a measureable function f so that $f_n \to f$ in measure. Furthermore, if $\{f_n\} \to f$ in measure, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ p.w.a.e. Also, if f_n also converges to some other function g in measure, then $f = g$ a.e.

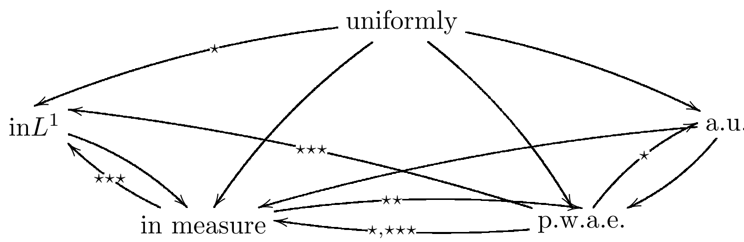
Theorem 3: If $f_n \to f$ in L^1, then there exists a subsequence f_{n_k} of f_n such that $f_{n_k} \to f$ a.e. For the proof of this result, merely combine the results of Theorem 1 and Theorem 2.

Definition: Let $f_n, f : X \to \mathbb{C}$ be $\mathcal{M}$-measureable functions. We say that $f_n \to f$ almost uniformly(a.u.) on X provided $\forall \epsilon > 0$, there exists an $E \in \mathcal{M}$ such that $\mu(E^c) \leq \epsilon$ and with $f_n \to f$ uniformly on E.

Egoroff's Theorem (Theorem 4): Assume that $\mu(X) \leq infty$, and assume that $f_n, f : X \to \mathbb{C}$ are $\mathcal{M}$- measureable such that $f_n \to f$ a.e. Then $f_n \to f$ almost uniformly.

Theorem 5: If $f_n \to f$ a.u., then $f_n \to f$ a.e. and in measure.

Illustration of what is going on here, along with certain facts from Homework:



Note the following conditions on the arrows:

1. $\star : \mu(X) < \infty$.
2. $\star\star$: A subsequence of the original sequence satisfies the conclusion.
3. $\star\star\star$: If the similar hypotheses to those of the DCT ($|f_n| \leq g \in L^1$) are satisfied, these conclusions are satisfied as well.

Product σ-algebras and measures on them

Definition: Let $\{X_1, \ldots, X_n\}$ be a finite collection of nonempty sets. Let $X = \prod_{j=1}^{n} X_j$ be their cartesian product and $\pi_j : X \to X_j$ be the jth projection. Let $\mathcal{A}_i$ be a σ-algebra on X_i for all i. We define the product σ-algebra on X to be the σ-algebra generated by the following collection of sets:

$$\{\pi_i^{-1}(E_i) | E_i \in \mathcal{A}_i,\ \forall E_i \in \mathcal{A}_i,\ \forall i\}$$

We denote the product σ-algebra by $\bigotimes_{i=1}^{n} \mathcal{A}_i = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. Note some basic facts:

1. $\pi_i^{-1}(E_i) = X_1 \times \cdots X_{i-1} \times E_i \times X_{i+1} \times \cdots \times X_n$
2. $\prod_j E_j = \bigcap_j \pi_j^{-1}(E_j) \in \bigotimes_i \mathcal{A}_i \ \ \forall E_i \in \mathcal{A}_i \ \forall i$
3. Note that the above (together with an even more trivial statement of the other containment) says that σ-algebra generated by the sets $\prod_i E_i$ is the same as $\bigotimes_i \mathcal{A}_i$, i.e. $\mathcal{M}(\{\prod_j E_j\}) = \bigotimes_j \mathcal{A}_j$.

Theorem: Let $X_1, \ldots, X_n$ be metric spaces and $X = X_1 \times \cdots \times X_n$ be the metric space endowed with the product metric. Then we have the following conclusions:

1. $\bigotimes_i \mathcal{B}_{X_i} \subset \mathcal{B}_X$
2. The σ-algebras above are the same provided the X_i are separable.

Definition: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, and let $\mathcal{M} \otimes \mathcal{N}$ be the product σ- algebra on $X \times Y$. For every $A \in \mathcal{M}$ and $B \in \mathcal{N}$, we have that $A \times B \in \mathcal{M} \otimes \mathcal{N}$, and we call it a **measureable rectangle**. Let $\mathcal{R} = \{A \times B \mid A \in \mathcal{M},\ B \in \mathcal{N}\}$. Then it is immediate that $\mathcal{R}$ is an elementary family. Recall, elementary families are collections of sets that are closed under finite intersections and complements of sets in the family can be written as disjoint unions of elements in the family.

What we wish to do is to construct a measure on $\mathcal{M} \otimes \mathcal{N}$, using μ and ν in a meaningful way. So, with this in mind, all we need to do is find an algebra and define a premeasure on that algebra. So, let $\mathcal{A}$ be the collection of all finite disjoint unions of elements in $\mathcal{R}$. Then by an old theorem, $\mathcal{A}$ is an algebra, and $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$.

$A \times B \in \mathcal{A}$ where $A \times B = \bigcup_j A_j \times B_j$ where the union may be finite or countable. Then we have for all $x \in X$ and $y \in Y$, we have $\chi_A(x)\chi_B(y) = \chi_{A \times B}(x,y) = \chi_{\bigcup A_j \times B_j}(x,y) = \sum \chi_{A_j \times B_j}(x,y) = \sum \chi_{A_j}(x)\chi_{B_j}(y)$.

Then we have $\chi(B)(y)d\mu(A) = \int_X \chi_A(x)\chi_B(y)d\mu(x) = \sum_j \int_X \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) = \sum_j \chi_{B_j}\mu(A_j)$, and then integrating the ends of the above equality we get $\nu(B)\mu(A) = \sum \nu(B_j)\mu(A_j)$, which is valid for all $A \times B \in \mathcal{A}$ such that $A \times B = \bigcup_j A_j \times B_j$, disjoint.

Next, we define $\pi : \mathcal{A} \to [0, \infty]$ by $\pi(E) = \sum_i \mu(A_j)\nu(B_j)$ with the convention in mind that $0 \cdot \infty = 0$. Then π is a premeasure on $\mathcal{A}$ as was just verified. By an old theorem, π generates an outer measure π^* on $\mathcal{P}(X \times Y)$, and $\pi^*|_{\mathcal{M} \otimes \mathcal{N}} := \lambda$ is a measure that extends π. This measure λ is called the product measure of μ and ν, and we denote it by $\lambda := \pi^*|_{\mathcal{M} \otimes \mathcal{N}} = \mu \times \nu$. Note that if μ and ν are σ-finite, then π is also σ-finite also. Hence, $\lambda = \mu \times \nu$ is the unique extension of π to $\mathcal{M} \otimes \mathcal{N}$ in this case.

Definition: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$be measure spaces and let $E \subset X \times Y$. For $x \in X$ and $y \in Y$, we define the x**-section** E_x and the y**-section** E^y by:

$$E_x := \{y \in Y \mid (x,y) \in E\}$$

$$E_y := \{x \in X \mid (x,y) \in E\}$$

Furthermore, given $f : X \times Y \to \mathbb{C}$ be a function. We defite the x**-section of** f, f_x, and y**-section of** f, f^y, by:

$$\begin{array}{ll} f_x : Y \to \mathbb{C} & f^y : X \to \mathbb{C} \\ y \mapsto f(x,y) & x \mapsto f(x,y) \end{array}$$

Note that the following is immediate for $E \subset X \times Y$: $(\chi_E)_x(y) = \chi_{E_x}(y)$ and $(\chi_E)^y(x) = \chi_{E^y}(x)$.

Theorem 1 on Product Measures: Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, and $\mathcal{M} \otimes \mathcal{N}$ the product σ-algebra. Then:

1. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E_y \in \mathcal{M}$ for all $x \in X$ and $y \in Y$.
2. Let $f : X \times Y \to \mathbb{C}$ be $(\mathcal{M} \times \mathcal{N})$-measureable. Then f_x is $\mathcal{N}$-measureable and f^y is $\mathcal{M}$-measureable.

Proof: The method of proof is as follows: Define $\mathcal{B} := \{E \subset X \times Y \mid E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}\ \forall x \in X, y \in Y\}$. Show that $\mathcal{R}$(defined above) is contained in $\mathcal{B}$, and show that $\mathcal{B}$ is a σ-algebra. Then $\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R}) \subset \mathcal{B}$, so elements of $\mathcal{M} \otimes \mathcal{N}$ satisfy the conclusions, hence the theorem holds.

Definition: Let $X \neq \emptyset$ and $\mathcal{C} \subset \mathcal{P}(X)$. We say that $\mathcal{C}$ is a **monotone class** on X provided $\mathcal{C}$ is closed under increasing countable unions and closed under decreasing countable intersections. I.e., if $\{E_j\} \subset \mathcal{C}$ with $E_j \subseteq E_{j+1}$ (resp. $E_j \supseteq E_{j+1}$) then $\bigcup_j E_j \in \mathcal{C}$ (resp. $\bigcap_j E_j \in \mathcal{C}$).

Note that any σ-algebra is a monotone class. Also, arbitrary intersections of monotone classes are also monotone classes. Hence given any set $\mathcal{E} \subset \mathcal{P}(X)$, we may define the **monotone class genereated by** $\mathcal{E}$, denoted $\mathcal{MC}(\mathcal{E})$, to be the smallest monotone class containing $\mathcal{E}$, i.e., the intersection of all monotone classes containing $\mathcal{E}$.

Lemma(Monotone Class Lemma): Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra on X. Then $\mathcal{MC}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

Theorem 2 On Product Measures: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in M \otimes N$, then we have the following:

1. $\alpha(x) := \nu(E_x)$ and $\beta(x) := \mu(E^y)$ are $\mathcal{M}$ and $\mathcal{N}$ respectively.
2. $\lambda(E) = (\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.

Theorem 3 On Product Measures (The Fubini-Tonelli Theorem): Let $(X, \mathcal{M}, \nu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Then

1. (Tonelli) Let $L^+(X \times Y)$ be the similar set for $\mathcal{M} \otimes \mathcal{N}$. Then for $f \in L^+(X \times Y)$, we have:
   (a) $g(x) := \int_Y f_x(y) d\nu(y) \in L^+(X)$ and $h(y) := \int_X f^y(x) d\mu(x) \in L^+(y)$.
   (b) $\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y)$
2. (Fubini) If $L^1(X \times Y, \mu \times \nu)$, then
   (a) $f_x \in L^1(\nu)$ for almost everywhere $x \in X$.
   (b) $f^y \in L^1(\mu)$ for almost everywhere $y \in Y$.
   (c) The almost everywhere defined functions $g(x) = \int_Y f_x(y) d\nu(y)$, $h(y) = \int_X f^y(x) d\mu(x)$ are $L^1(\mu)$ and $L^1(\nu)$ respectively.
   (d) $\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y)$.