1. Let $R$ be a Noetherian domain and $M$ a finitely generated $R$-module. Prove that the set of primes $p$ of $R$ such that $M_p$ is torsion free is an open subset of $\text{Spec } R$.

**Proof:** Claim: $T(M)_S = T(M_S)$ where $T(\cdot)$ denotes the torsion submodule. Let $\frac{m}{s} \in T(M)_S$. Then $\exists t \in R$ a nonzerodivisor such that $tm = 0$. Thus, $\frac{t}{1} \cdot \frac{m}{s} = 0 = \frac{0}{s}$. Note that $\frac{t}{1}$ is a nonzerodivisor as $t$ is a nonzerodivisor. Indeed, if $\frac{1}{T} \neq \frac{1}{T}$ then there would exist a $k \in S$ such that $ktu = 0$, a contradiction. Hence, $\frac{m}{s} \in T(M_S)$. Next, let $\frac{m}{s} \in T(M_S)$. Then $\exists t$ a nonzerodivisor such that $\frac{tm}{s} = 0 = \frac{0}{s}$. So, there exists $k \in S$ such that $krm = 0$. Now $kr$ is not a zerodivisor for similar reasons above, so that $m \in T(M)$. Hence, $\frac{m}{s} \in T(M_S)$. Now, (suggestively) let $\mathcal{O} = \{p \in \text{Spec } R \mid M_p$ is torsion free$\}$. So, we know that $p \in \mathcal{O} \iff T(M_p) = 0 \iff T(M)_p = 0$, by the claim above. So, since $T(M)_p = 0$ and $T(M)$ is finitely generated (since its a submodule of a finitely generated module over a Noetherian ring), we have that $T(M)_p = 0 \iff \text{Ann}_R T(M) \notin p$. So, $p \in \mathcal{O} \iff \text{Ann}_R T(M) \notin p$, hence we have by definition that $\mathcal{O} = \text{Spec } R \setminus \text{Ann}_R T(M)$, hence $\mathcal{O}$ is open in $\text{Spec } R$.

2. Let $R$ be a domain. Suppose $0 \to P_1 \to P_0 \to M \to 0$ is an exact sequence of $R$-modules where $P_1$ and $P_0$ are finitely generated projective $R$-modules. Prove that rank $P_1 \leq$ rank $P_0$ with equality if and only if $M$ is torsion.

**Proof:** As $R$ is a domain, the ideal $(0)$ is prime. So we will localize the above exact sequence at $(0)$.

Also, note that a domain localized at the zero ideal is the field of fractions $Q(R)$ of $R$. Also, as $P_1$ and $P_0$ are finitely generated projective $R$-modules they are locally free, so that $(P_1)_0 \cong (Q(R))^m$ and $(P_0)_0 \cong (Q(R))^n$ where $m = \text{rank } P_1$ and $n = \text{rank } P_0$. So, localizing the above exact sequence gives the exact sequence $0 \to (Q(R))^m \xrightarrow{f} (Q(R))^n \to M_0 \to 0$. So, the map $f$ is injective by exactness, and is a map of $Q(R)$-vector spaces. Therefore, we must have that $m \leq n$. In addition, $m = n$ if and only if $f$ is an isomorphism if and only if $M_0 = 0$. So if $M_0 = 0$, then $\frac{m}{s} = \frac{0}{1}$ for all $\frac{m}{s} \in M_0$. So, we have for each $m \in M$, there exists a $t \neq 0$ such that $tm = 0$. Note all nonzero ring elements of $R$ are nonzerodivisors. Thus $M$ is torsion. Suppose $M$ is torsion, and let $\frac{m}{s} \in M_0$. Then as $M$ is torsion, there exists a nonzerodivisor $t$ (i.e. $t \notin \mathcal{O}$) such that $tm = 0$. Therefore, $\frac{m}{s} = \frac{0}{1}$ for all $\frac{m}{s} \in M_0$. Therefore $M_0 = 0$. So, $m = n$ if and only if $M$ is torsion.

3. A ring is called reduced if it has no nonzero nilpotent elements. Prove that $R$ is reduced if and only if $R_p$ is reduced for all $p \in \text{Spec } R$.

**Proof:** $\Rightarrow$: Let $R$ be reduced, and let $\frac{r}{s} \in R_p$ such that $(\frac{r}{s})^k = 0$ for $p \in \text{Spec } R$. Then $\frac{r^k}{s^k} = 0 = \frac{0}{1}$ so that there exists $t \in R \setminus p$ such that $tr^k = 0$. But, we note that $(tr)^k = t^k r^k = t^{k-1} tr^k = 0$. So, as $R$ is reduced, we know that $tr = 0$. So, we have that $\frac{r}{s} = \frac{0}{1}$. Hence, $R_p$ is reduced for all $p \in \text{Spec } R$.

$\Leftarrow$: Let $R_p$ be reduced for all $p \in \text{Spec } R$. Let $r \in R$ be such that $r^k = 0$ for some $k$. Then $(\frac{r}{s})^k = \frac{r^k}{s^k} = 0 = \frac{0}{1}$. Hence $\frac{r}{s} = \frac{0}{1}$ in $R_p$ for all $p \in \text{Spec } R$ as $R_p$ is reduced. So, for all $p \in \text{Spec } R$, there exists $t \in R \setminus p$ such that $tr = 0$. In particular, for all maximal ideals $m$, there exists $t \in R \setminus m$ such that $tr = 0$. So, let $m^* \in \mathcal{O}$ be the maximal ideal containing $\text{Ann}_R r$ (if $\text{Ann}_R r = R$ then $r = 0$, so then we are done). So by the above argument, there exists a $t \notin m^*$ such that $tr = 0$, a contradiction to $t \notin \text{Ann}_R r$. Hence $r = 0$. Therefore, $R$ is reduced.
4. Let \( R \) be a Noetherian ring. Prove that there exists a nonzero element \( x \in R \) such that \( \text{Ann}_R x \) is a prime ideal of \( R \).

**Proof:** Suppose that for all nonzero \( x \in R \), we have that \( \text{Ann}_R x \) is not a prime ideal, and let \( x \in R \) such that \( \text{Ann}_R x = R \), which we may do in any nonzero ring with identity. Then there exist \( a_1, r_1 \notin \text{Ann}_R x \) such that \( a_1 r_1 \in \text{Ann}_R x \). Now we also have that \( \text{Ann}_R x \) is not prime by assumption so that there exist \( a_2, r_2 \notin \text{Ann}_R x \) such that \( a_2 r_2 \in \text{Ann}_R x \), and we can also consider \( \text{Ann}_R x \). Note that \( \text{Ann}_R x \subseteq \text{Ann}_R x \subseteq \text{Ann}_R x \) as \( a_2 \notin \text{Ann}_R x \) but not in \( \text{Ann}_R x \) and \( a_1 \in \text{Ann}_R x \) but not in \( \text{Ann}_R x \). Continuing in this manner, we may construct an infinite ascending chain of ideals, contradicting that \( R \) is Noetherian. Therefore, there must exist a spot in our chain where this process stops, namely, an \( x \) such that \( \text{Ann}_R x \) is prime.

5. Let \((R, \mathfrak{m})\) be a quasi-local ring and \( f : R^n \to M \) a surjective \( R \)-module homomorphism. Prove that \( n = \mu_R(M) \) if and only if \( \ker f \subseteq \mathfrak{m} R^n \).

**Proof:** First we establish the following general result: Let \( f : M \to N \) be a surjective \( R \)-module homomorphism and \( I \) an ideal of \( R \). Then the natural induced map \( \bar{f} : M/IM \to N/IN \) is an isomorphism if and only if \( \ker f \subseteq IM \).

\[ \Rightarrow : \text{As } f : M \to N \text{ is surjective, we know that } \bar{f} \text{ is. Also, let } m \in M \text{ such that } f(m) = 0. \text{ Then } f(\bar{m}) = f(m) + IN = 0, \text{ so that since } f \text{ is 1-1, we have that } \bar{m} = 0, \text{ thus } m \in IM. \]

\[ \leftarrow : \text{Let } \bar{m} = m + IM \text{ be such that } f(\bar{m}) = 0. \text{ Then } f(m) \in IN. \text{ Hence we have that } f(m) = \sum i_j m_j \text{ for some } i_j \in I \text{ and } n_j \in N. \text{ As } f \text{ is surjective for each } n_j \text{ there exists } m_j \text{ such that } f(m_j) = n_j. \text{ So, } f(m) = \sum i_j f(m_j) = f(\sum i_j m_j). \text{ So, as } f \text{ is a homomorphism, we have that } f(m - \sum i_j m_j) = 0, \text{ so that } m - \sum i_j m_j \in ker f. \text{ Therefore by assumption, we have that } m - \sum i_j m_j \in IM, \text{ so that } m \in IM. \text{ Therefore, } \bar{m} = 0, \text{ so that } f \text{ is injective. } f \text{ is clearly surjective, as } f : M \to N \text{ is surjective. This completes the proof of the claim.} \]

So, applying this to the situation above, we have that if \( f : R^n \to M \) is a surjective \( R \)-module homomorphism, then \( \bar{f} : (R/\mathfrak{m})^n \to M/\mathfrak{m} M \) is an isomorphism if and only if \( \ker f \subseteq \mathfrak{m} R^n \). Note that in this case, we have that \( \bar{f} \) is a map of \( R/\mathfrak{m} \) vector spaces so that the images of the generators of \((R/\mathfrak{m})^n\) are generators for \( M/\mathfrak{m} M \), say \( \{\bar{x}_1, \ldots, \bar{x}_n\} \). Thus, by NAK, \( x_1, \ldots, x_n \) generate \( M \), so that \( \mu_R(M) = n \).

6. Let \( E/F \) be a field extension and \( S = \{x_1, x_2, x_3\} \) a subset of \( E \) which is algebraically independent over \( F \). Prove that \( T = \{x_1 + x_2, x_1 x_2 + x_1 x_3 + x_2 x_3, x_1 x_2 x_3\} \) is algebraically independent over \( F \).

**Proof:** Note that \( T \) consists of precisely the elementary symmetric polynomials over the algebraically independent set \( \{x_1, x_2, x_3\} \). Thus by a theorem in class, we have that \( F(S)/F(T) \) is Galois, and hence algebraic. Also, note that we have that \( \text{trdeg}(F(S)/F) = 3 \), so that \( \text{trdeg}(F(T)/F) \) is at most 3. Furthermore, we know that \( \text{trdeg}(F(S)/F) = \text{trdeg}(F(S)/F(T) + \text{trdeg}(F(T)/F) = \text{trdeg}(F(T)/F) \) as \( \text{trdeg}(F(S)/F(T) = 0 \) because \( F(S)/F(T) \) is algebraic. So, \( \text{trdeg}(F(T)/F) = 3 \). By a HW problem, since \( F(S) \) is algebraic over \( F(T) \), \( T \) contains a transcendence base for \( F(S)/F \). Therefore, \( T \) must be a transcendence base for \( F(S)/F \) hence algebraically independent.

7. Let \( R \) be a Noetherian domain in which every ideal is projective. Prove that \( \dim R \leq 1 \).

**Proof:** Let \( I \) be an ideal of \( R \). Then \( I \) is a finitely generated projective \( R \)-module by assumption, and since \( R \) is Noetherian. Also, for all \( p \in \text{Spec } R \), \( I_p \) is a free \( R_p \)-module because f.g. projectives are locally free. Note that all ideals of \( R_p \) are of the form \( I_p \) for some ideal \( I \) of \( R \). So, we have that
every ideal of $R_p$ is free. By a HW problem, we have that every ideal is principally generated by a nonzerodivisor, so that $R_p$ is a PID for all $p \in \text{Spec } R$. I claim that if $R$ is a PID then $\dim R \leq 1$. Indeed, suppose there existed a chain of proper prime ideals $(0) \subsetneq (f_1) \subsetneq (f_2)$. Then as $f_1 \in (f_2)$ there exists an $r \in R$ such that $rf_2 = f_1$. However, as $(f_1)$ is prime, $r \in (f_1)$, so there exists $s \in R$ such that $sf_1 = r$. So we have that $sf_1f_2 = f_1 \Rightarrow f_1(sf_2 - 1) = 0 \Rightarrow sf_2 = 1 \Rightarrow f_2$ is a unit, a contradiction to the fact that $(f_2)$ was a prime ideal. So, $\dim R_p \leq 1$ for all $p \in \text{Spec } R$. Note that $\dim R$ may be defined as:

$$\dim R = \sup \{ \dim R_p \mid p \in \text{Spec } R \}.$$ 

To see this, note that the dimension of a ring $R$ is the sup of the lengths of chains of prime ideals of $R$. Also, the prime ideals of $R_p$ are just the prime ideals contained in $p$. So, taking the dimension of the localizations over all prime ideals is the same as looking for the longest chain of prime ideals in the original ring. Therefore, as $\sup \{ \dim R_p \mid p \in \text{Spec } R \} \leq 1$ we have that $\dim R \leq 1$ as desired.