1. Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a short exact sequence of $R$-modules and $I$ an ideal of $R$. Prove that $L/IL \xrightarrow{f} M/IM \xrightarrow{g} N/IN \rightarrow 0$ (with the natural induced maps) is exact. Give an example to show the map $L/IL \rightarrow M/IM$ does not need to be injective.

**Proof:** We must show that $\tilde{g}$ is surjective and that $\text{im} \tilde{f} = \ker \tilde{g}$. First of all, let $\bar{n} = n + IN$ be an element of $N/IN$. Then as $g$ is surjective, there exists an $m \in M$ such that $g(m) = n$. Let $\bar{m}$ be the coset of $m$ in $M/IM$. Then $\tilde{g}(\bar{m}) = g(m) + IN = n + IN = \bar{n}$. This proves surjectivity of $\tilde{g}$.

To show exactness at $M/IM$, let $m + IM = \bar{m} \in \text{im} \tilde{f}$. Then $\exists \ \bar{l} = l + IL$ such that $\tilde{f}(\bar{l}) = f(l) + IM = m + IM$. However, $\tilde{g}(\bar{m}) = \tilde{g}(\tilde{f}(\bar{l})) = g(f(l)) + IN = 0 + IN$, by exactness of the original exact sequence. Therefore $\bar{m} \in \ker \tilde{g}$.

Conversely, let $\bar{m} \in \ker \tilde{g}$. Then $\tilde{g}(\bar{m}) = g(m) + IN = 0 + IN$. Therefore, $g(m) \in IN$. So, write $g(m) = \sum_{j=1}^{n} i_j m_j$ where $i_j \in I$, $m_j \in N$. Now, as $g$ is surjective, $\exists \ m_j \in M$ such that $g(m_j) = n_j$ for all $j = 1, \ldots, n$. Now we have that $g(m) = \sum_{j=1}^{n} i_j m_j = \sum_{j=1}^{n} i_j g(m_j) = g(\sum_{j=1}^{n} i_j m_j)$

Therefore, we have that $g(m - \sum_{j=1}^{n} i_j m_j) = 0$, therefore $m - \sum_{j=1}^{n} i_j m_j \in \ker g$. So, by exactness of the original sequence, we have that there exists $l \in L$ such that $f(l) = m - \sum_{j=1}^{n} i_j m_j$. Hence $f(\bar{l}) = f(l) + IM = m - \sum_{j=1}^{n} i_j m_j + IM = m + IM = \bar{m}$. Therefore, we have that $\bar{m} \in \text{im} \tilde{f}$.

To see that the map $L/IL \rightarrow M/IM$ need not be injective, consider the polynomial ring $R = \mathbb{Z}[x]$, and take $L = M = R$. Then the map $L \xrightarrow{\tilde{g}} M$ is injective, but if we let $I = (x^2)$, and consider the induced map $L/IL \xrightarrow{\tilde{g}} M/IM$, we notice that the coset $\bar{x}$ is in the kernel of the new map (because $\bar{x}$ is no longer a nonzerodivisor). Therefore, the map is no longer injective.

2. Let $S = R[[x]]$ be the ring of formal power series in one variable over $R$. Prove that $g = g_0 + g_1 x + g_2 x^2 + \cdots \in S$ is a unit if and only if $g_0$ is a unit in $R$. Use this to prove that if $R$ is local, then so is $R[[x]]$.

**Proof:**

"$\Rightarrow$": Let $g = g_0 + g_1 x + g_2 x^2 + \cdots \in S$ be a unit. Then $\exists h = h_0 + h_1 x + h_2 x^2 + \cdots \in S$ such that $gh = 1$. In other words, $g_0 h_0 + g_1 h_0 x + g_2 h_0 x^2 + \cdots + g_1 x h_0 + g_1 h_1 x + h_2 x^2 + \cdots + 1 = 1$. Therefore, all coefficients of the powers of $x$ must cancel, leaving the only constant term, $g_0 h_0 = 1$. Therefore, $g_0 \in R$ is a unit (with inverse $h_0$).

"$\Leftarrow$": Let $g = g_0 + g_1 x + \cdots \in S$ with $g_0$ a unit in $R$. Then $\exists h_0$ such that $g_0 h_0 = 1$. Note that $g$ is a unit if and only if $g' = 1 + g_1 x + \cdots$ is a unit where $g'_1 = g_1 / h_0$. Define $h' = 1 + h'_1 x + \cdots$ where the $h'_i$ are chosen so as to cancel the extra higher powers of $x^i$ off. For example, choose $h'_1 = -g'_1$, $h'_2 = g'_1^2 - g'_2$, $h'_3 = 2g'_1 g'_2 - g'_3 + g'_4$, and so on (Tom, I couldn’t for the life of me find a pattern to the coefficients involved, but I think its clear that formally you may always find the coefficient left over when multiplying by the previous terms). Performing this process forever, we have created an element that when multiplied by $g'$ gives 1. Therefore $g'$ is a unit. Hence $g$ is also a unit.

For the final conclusion, Define $m = \{ f \in S | f \text{ has a non-unit constant term} \}$. Then $m$ forms an ideal, as $R$ is local, and by the last problem of homework 1. However, by the proposition, $m$ is exactly the set of non-units in $S$, hence the set of non-units of $S$ form an ideal, so $S$ is also local.
3. Let \( R[x] \) be a polynomial ring in one variable over \( R \) and let \( I \) be an ideal of \( R \). Prove that \( R[x]/IR[x] \cong (R/I)[x] \). Conclude that if \( P \) is a prime ideal of \( R \) then \( PR[x] \) is a prime ideal of \( R[x] \).

**Proof:** Define
\[
\phi : R[x] \to (R/I)[x]
\]
\[
f(x) \mapsto \tilde{f}(x)
\]
where if \( f = a_0 + a_1 x + \cdots + a_n x^n \) then \( \tilde{f} = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n \), where \( \bar{a}_i \) denotes the coset of \( a_i \) in \( R/I \). This is clearly a well-defined ring homomorphism. I claim that \( \ker \phi = IR[x] \). Now \( IR[x] = \{ f \in R[x] \mid f = a_0 + a_1 x + \cdots + a_n x^n \text{ and } a_i \in I \ \forall i = 1, \ldots, n \} \) Clearly we have that \( IR[x] \subseteq \ker \phi \). Let \( f \in \ker \phi \). Then \( \tilde{f} = 0 \), i.e. \( \bar{a}_i = 0 \ \forall i = 1, \ldots, n \). Hence we also have \( \ker \phi \subseteq IR[x] \). Hence in the case that \( P \) is a prime ideal of \( R \), then \( (R/P)[x] \) is a domain, so \( (R/P)[x] \) is a domain, hence \( R[x]/PR[x] \) is a domain by the previous isomorphism, and thus \( PR[x] \) is a prime ideal of \( R[x] \).

4. Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x] \). Prove that \( f(x) \) is a unit if and only if for all \( a_i \) is a unit and \( a_i \) is nilpotent for all \( i > 0 \). (Hint: For one direction, you can prove that a unit plus a nilpotent is always a unit. For the other direction, first prove it in the case \( R \) is a domain and then use the previous exercise.)

**Proof:** "\( \Rightarrow \):" Let \( a_0 \) be a unit, and \( a_i \) be nilpotent for \( i = 1, \ldots, n \). First I will prove a claim. I claim that if \( u \) is a unit and \( n \) is nilpotent, then \( u + n \) is a unit.

**Proof of Claim:** Since \( n \) is nilpotent, \( n^k = 0 \) for some \( k \) (where we take \( k \) to be the smallest integer such that \( n^k = 0 \)). Also, as \( u \) is a unit, \( uu^{-1} = 1 \) for some \( u^{-1} \in R \). Now note that
\[
1 = (u^{-1})^k(u^k + n^k) = (u^{-1})^k(u^k - u^{k-2} + \cdots + (-1)^{k-2}u^{k-2} + (-1)^{k-1}n^{k-1})(u + n)
\]
Hence, we have that \( (u^{-1})^k(u^k - u^{k-2} + \cdots + (-1)^{k-2}u^{k-2} + (-1)^{k-1}n^{k-1}) = (u + n)^{-1} \), thus \( (u + n) \) is a unit. This proves the claim.

Therefore, if \( a_i \) is nilpotent, then \( a_i^{k_i} = 0 \) for some \( k \), therefore, \( (a_i x^i)^k = 0 \). So, \( a_i x^i \) is nilpotent for all \( i > 0 \). Hence, \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) is a unit.

"\( \Leftarrow \):" For the other direction, we will prove it in the case that \( R \) is a domain and reduce the general case to the domain case.

So, assume \( R \) is a domain, and let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x] \) be a unit. Then \( \exists g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x] \) such that \( f(x)g(x) = 1 \). Therefore, we must have that \( a_n b_m = 0 \) as they are the only contributing coefficients to that power of \( x \). Therefore, as \( R \) is a domain, we must have that \( a_n \) or \( b_m \) must be zero. Repeating this argument for all powers of \( x \), we get that \( a_i = b_i = 0 \) for all \( i > 0 \). Therefore, the proposition holds in the case of a domain, as the only nilpotent element in a domain is zero.

Now, assume \( R \) is a commutative ring with identity. Let \( f \in R[x] \) as above be a unit. Then \( \exists g(x) \in R[x] \) such that \( f(x)g(x) = 1 \). Now let \( P \) be a prime ideal. Sending \( f \) and \( g \) to \( (R/P)[x] \), we know that \( f(x)\tilde{g}(x) = 1 \) (where the tildes mean the same thing as in problem 3), as \( 1 \not\in P \). But \( R/P \) is a domain, so \( \tilde{a}_i = \bar{b}_i = 0 \forall i > 0 \). Therefore, \( a_i \in P \), and \( b_i \in P \). Since \( P \) was arbitrary, by Krull’s theorem we see that \( a_i \) in nilpotent for all \( i > 0 \). Clearly, we see that \( a_0 \) is a unit, as \( a_0 b_0 = 1 \), with this being the only part in the product of \( f \) and \( g \) contributing to the constant part of the result.
5. Prove that free objects on nonempty sets do not exist in the category of fields.

**Proof:** First note that a field map is either trivial or injective, hence a field map must either be zero or preserve the characteristic of the field (else how could we embed a field of characteristic 2 in a field of characteristic 0?). For a free field $F$ to exist on a nonempty set $X$, we require that there exist a set map $t : X \to F$ such that for any set map $f : X \to K$ where $K$ is any field, then there must exist a unique field map $g : F \to K$ such that $tg = f$. Now as non-zero field maps must preserve characteristic, if we are to have a non-zero map for $g$, we are requiring $F$ to simultaneously have every characteristic, which is absurd. Thus $g$ must be zero. However, this certainly restricts our choice of set map from $X$ to $K$, so we cannot build a free object on a non-empty set in the category of fields.

6. Let $I$ be an ideal of $R$ and $S$ a multiplicatively closed set of $R$. Let $\bar{S}$ denote the image of $S$ under the natural map $R \to R/I$. Then $\bar{S}$ is a multiplicatively closed set of $R/I$. Prove that $R_S/I_S \cong (R/I)_{\bar{S}}$ as rings. Conclude that if $p \in \text{Spec}(R)$ then $R_p/p_p$ is isomorphic to the quotient field of $R/p$.

**Proof:** Define

\[ \phi : R_S \to (R/I)_{\bar{S}} \]

\[ \frac{r}{s} \mapsto \frac{\bar{r}}{\bar{s}} \]

If $\phi$ is well defined, it is clearly a ring homomorphism. To check this, let $\frac{r_1}{s_1} = \frac{r_2}{s_2}$. Then $\exists t \in S$ such that $t(r_1s_2 - r_2s_1) = 0$. Therefore, we also have $\bar{t}(\bar{r}_1\bar{s}_2 - \bar{r}_2\bar{s}_1) = 0$. Hence, $\frac{r_1}{s_1} = \frac{r_2}{s_2}$, as $t \in \bar{S}$.

I claim that $\ker \phi = I_S$. Indeed, let $\frac{r}{s} \in \ker \phi$. Then $\frac{r}{s} = \frac{0}{1}$. Hence, $\exists \bar{t} \in \bar{S}$ such that $\bar{r}t = 0$, hence $rt \in I$. Therefore, we have that $\frac{r}{s} = \frac{rt}{s} \in I_S$. Now let $\frac{1}{s} \in I_S$. Then $\phi(\frac{1}{s}) = \frac{\bar{1}}{\bar{s}} = \frac{0}{\bar{s}} = 0$. Therefore, $\frac{1}{s} \in \ker \phi$.

Now let $p \in \text{Spec}(R)$. Then by the above, we have that $R_p/p_p \cong (R/p)_p$. Note that localizing $(R/p)$ at $p$ is equivalent to inverting all elements of $(R/p)$ that are not zero (not in $p$), which is the same construction of the quotient field of a domain (which $(R/p)$ is).

7. Let $M$ be a finitely generated $R$-module and $p \in \text{Spec}(R)$. Prove that $M_p = 0$ if and only if $\text{Ann}_RM \not\subset p$.

**Proof:** $\Rightarrow$: Choose $t \in \text{Ann}_RM \setminus p$. Now let $\frac{m}{t} \in M_p$. Then $\frac{m}{t} = \frac{0}{1}$, as $tm = 0$.

$\Leftarrow$: Let $m_1, \ldots, m_r$ generate $M$. We know that $\frac{m_i}{t} = 0$ in $M_p \ \forall i$. Therefore, $\exists t_i \in R \setminus p$ such that $t_im_i = 0$. Set $t = t_1t_2\cdots t_n$, which is in $R \setminus p$ as $p$ is prime. Now by construction, we have that $tm_i = 0 \ \forall i$, so $t \in \text{Ann}_RM \setminus p$. 

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