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Algebra 901 Notes
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Internal Semidirect Products:
Recall for direct products, we have that $G = HK, H \cap K = \{1\}$, and $H \lhd G$ and $K \lhd G$ together imply that $G = H \times K$.

Theorem/Definition: Let G be a group and suppose $\exists$ subgroups H, K such that $G = HK, H \cap K = \{1\}$, and $H \lhd G$. Then $\exists$ a homomorphism $\phi : K \to \mathrm{Aut}(H)$ such that $G \cong H \rtimes_\phi K$.

In this situation, G is the internal semi-direct product of H and K. A note first, the notion of internal and external semi-direct product is only a formal distinction, since if H and K are groups, you can consider them living inside $G := H \rtimes_\phi K$. So, this terminology is justified by the previous isomorphism (i.e. the direct product).

Recall, in light of the above comment, we talked about $G = H \rtimes_\phi K$, and we noted that $G = H'K', H' \lhd G$ and $H' \cap K' = \{1\}$, so we really have talked about this before.

Proof: Let $y \in K$ and consider the inner automorphism ψ_y on G:

$$\psi_y : G \to G$$

$$x \mapsto yxy^{-1}$$

So, $\psi_y(H) = yHy^{-1} = H$ as $H \lhd G$. So, $\psi_y|_H \in Aut(H)$. Define ϕ to be the following:

$$\phi : K \to \mathrm{Aut}(H)$$

$$y \to \psi_y|_H$$

This is clearly a group homomorphism. Now, define f to be the following:

$$f : H \rtimes_\phi G \to G$$

$$(h, k) \mapsto hk$$

Clearly this map is well defined, is 1-1 and onto by assumption on the conditions of H and K, so what is left to show is that f is a group homomorphism.

$$f((h_1, k_1)(h_2, k_2)) = f((h_1k_1h_2k_1^{-1}, k_1k_2))$$

$$= h_1k_1h_2k_1^{-1}k_1k_2$$

$$= h_1k_1h_2k_2$$

$$= f((h_1, k_1))f((h_2, k_2))$$

Therefore, f is an isomorphism.

Theorem: Let G be a group of order $2p$ where p is a prime bigger than 2. Then $G \cong C_{2p}$ or $G \cong D_{2p} \cong C_p \rtimes C_2$.

Proof: Let $P \in \mathrm{Syl}_2(G), Q \in \mathrm{Syl}_p(G)$. We know that $Q \lhd G$, as $[G : Q] = 2$, and $G = PQ$, and $P \cap Q = \{1\}$. So, $G = Q \rtimes_\phi P$ for some $\phi : Q \to \mathrm{Aut}(P)$. Let $P = \langle y \rangle$ where $o(y) = 2$. So, $\phi(y)$ has order 1 or 2 in $\mathrm{Aut}(Q)$. If $|\phi(y)| = 1$ then $\phi(y) = \mathrm{Id}_Q$, so $G = Q \times P \cong C_2 \times C_p \cong C_{2p}$. Now $\mathrm{Aut}(Q)$ is cyclic of order $p - 1$. Therefore $\exists$ exactly one element of order 2 in $\mathrm{Aut}(Q)$, given by inversion. So $G \cong D_{2p}$ by our previous construction. I.e., if $Q = \langle x \rangle$, then $yxy^{-1} = x^{-1}$ or $yx = x^{-1}y$, so G has the presentation

$$G = \langle x, y | x^p = 1, y^2 = 1, yx = x^{-1}y \rangle \cong D_{2p}.$$

Lemma(Theorem): Let K be a cyclic group of order n, H an arbitrary group, and let $\phi_1, \phi_2 : K \to \mathrm{Aut}(H)$ be group homomorphisms. Suppose $\phi_1(K)$ and $\phi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$. Then $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$. Important special cases are when $\phi_1(K) = \phi_2(K)$ or when $\phi_1(K)$ and $\phi_2(K)$ are Sylow p-subgroups.
Proof:???

Major Example I: Classify all groups up to order 12. So, let $P \in \mathrm{Syl}_2(G)$ and $Q \in \mathrm{Syl}_3(G)$. By a homework problem(on groups of order p^2q, either P or Q is normal. Note first that we have 2 abelian ones, namely C_{12} and $C_2 \times C_2 \times C_3 \cong C_2 \times C_6$. We will break the nonabelian argument into cases.

1. $P \lhd G$. Therefore, $P \rtimes_\phi Q$ for some $\phi : Q \to \mathrm{Aut}(P)$.
    (a) Suppose $P \cong C_4$. So $\mathrm{Aut} P \cong \mathbb{Z}_4^* \cong C_2$. Since $|Q| = 3$, $\phi : Q \to \mathrm{Aut}(P)$ mus be trivial, so no new nonabelian groups.
    (b) Suppose $P \cong C_2 \times C_2$. So $\mathrm{Aut}(P) \cong GL_2(\mathbb{Z}_2)$, and $|GL_2(Z_2)| = 6$. Thus, if ϕ sends Q to a subgroup of order 3 in $\mathrm{Aut}(P)$, we get a nontrivial homomorphism and since $\phi(Q)$ is a sylow 3-subgroup, we see there is only one non-abelian group in this case, namely $(C_2 \times C_2) \rtimes_\phi C_3$. Note that this is A_4.
2. $Q \lhd G$. So $G = Q \rtimes_\phi P$ and $\mathrm{Aut}(Q) \cong Z_3^*$. So $phi : P \to \mathrm{Aut}(Q)$.
    (a) Say $P \cong C_4$, and set $P = \langle x \rangle$ and $Q = \langle y \rangle$. Define $\phi : P \to \mathrm{Aut}(Q)$ by sending x to the inversion map on Q. Therefore, conjugation by an element from P inverts an element in Q. So, a presentation is: $\langle x, y \mid x^4 = 1, y^3 = 1, xyx^{-1} = y^{-1} \rangle$. This is not isomorphic to A_4 since the sylow 2-subgroup is not normal.
    (b) Suppose $P \cong C_2 \times C_2$. Suppose that $\phi : P \to \mathrm{Aut}(Q)$ is nontrivial. Then ϕ must be surjective as $|\mathrm{Aut}(Q)| = 2$. Therefore, $|\ker \phi| = 2$. Let $\ker \phi = \langle a \rangle$. Then $\phi(a) = \mathrm{Id}_Q$. But $\phi(a)$ really is conjugation by a, so $aqa^{-1} = q$ for all $q \in Q$. Let $H = \langle a \rangle$. Then HQ is a subgroup of G as $Q \lhd G$. Also, we have that $|HQ| = 6$, hence $HQ \lhd G$ by indexes. Also, since $o(a) = 2$ and $o(y) = 3$ and $ay = ya$, $o(ay) = 6$ so that $HQ \cong C_6$. Let $b \notin \ker \phi$.