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Algebra 901 Notes
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Example: Let G be a group of order $3 \cdot 5 \cdot 7 = 105$. Prove that the Sylow 5 and 7 subgroups are normal.Moreover, the sylow 5 subgroup is contained in the center of thegroup.
Proof: Suppose $n_5 > 1$ and $n_7 > 1$. Then$n_5 = 21$ and $n_7 = 15$. This gives 84 non-identity elements inthe sylow 5 subgroups and 15*6 non-identity elements in the sylow7-subgroups. This is much bigger than 105 elements, so weconclude that either the sylow 5 subgroup or the sylow 7 subgroupis normal. Let $P \in \text{Syl}_5(G)$ and $Q \in \text{Syl}_7(G)$. ThenPQ is a subgroup of order 35. Also notice that $[G : PQ] = 3$,so $PQ \lhd G$ since 3 is the smallest prime dividing the order of$G$. Also, PQ is cyclic, since $5 \nmid 7 - 1$. Let $P' \in \text{Syl}_5(G)$. Then $P' = yPy^{-1}$ for some $y \in G$, and$pPy^{-1} \subseteq yPQy^{-1} = PQ$. So P' is a sylow 5subgroup of PQ. Since P and P' are both Sylow 5-subgroupsof PQ and PQ is abelian (cyclic), then $P = P'$. Hence $n_5 = 1$. The exact same argument works for $n_7 = 1$. For the laststatement, we recall a lemma and give a definition.

Recall a Lemma: Let H, K be subgroups of G,and suppose that H, K are normal in G and that $H \cap K = \{1\}$. Then $hk = kh \;\; \forall h \in H$ and $k \in K$.
Proof: Note that $hkh^{-1}k^{-1} \in H \cap K = \{1\}$.

Definition: Let H be a subgroup of G.Denote the centralizer of H in G, $C_G(H) := \{\, g \in G | gh = hg \; \forall h \in H\}$. Note that this is a subgroup and$C_G(H) = G \Leftrightarrow H \subseteq Z(G)$.

Proof of $P \subseteq Z(G)$: Since P is cyclic, we have $P \subseteq C_G(P)$. Also, since PQ is cyclic,we have $Q \subseteq C_G(P)$. Let $R \in \text{Syl}_3(G)$. As P is normal, PR is a subgroup of G of order 15 and hence cyclic.Therefore, $R \subseteq C_G(P)$. Therefore, $G = PQR \subseteq C_G(P) \;\Rightarrow P \subseteq Z(G)$.

Definition: A **simple group** is a group$\neq \{1\}$, and whose only normal subgroups are 1 and itself.As an example, groups of prime order are simple.

Example: Prove that any group of order $144 = 2^4 \cdot 3^2$ is not simple.
Proof: We see that $n_3 = 1, 4$ or 16. So, let$P \in \text{Syl}_2(G)$, $Q \in \text{Syl}_3(G)$. We break the argument up into cases.

1. $n_3 = 1$. We're done, because then the sylow 3-subgroup isnormal.

2. Suppose $n_3 = 4$. Recall $n_3 = [G : N_G(Q)] = 4$. By the Cayley map, we see that $\phi : G \to \text{Perm}(G/N_G(Q)) \cong S_4$. Since the order of G is less than the order of S_4, we see we have a homomorphism with a non-trivial kernel. Since $\ker\phi$ is nontrivial and normal in G, G is not simple.

3. Suppose $n_3 = 16$. Let $Syl3(G) = \{Q_1, \ldots Q_16\}$.We break this argument into cases as well.

   (a) $Q_i \cap Q_j = \{1\} \quad \forall i \neq j$. Then there are$16 \cdot 8 = 128$ non-identity elements on the sylow 3-subgroups,and there are 16 elements left to form **the** sylow 2-subgroup, and therefore its normal, causing G to not besimple.

   (b) $|Q_i \cap Q_j| = 3$ for some $i \neq j$. Reorder so thatwe have $|Q_1 \cap Q_2| = 3$. Consider $N = N_G(Q_1 \cap Q_2)$.Since $|Q_i| = 3^2$, we know that Q_i is abelian for all i,and therefore $Q_1 \cap Q_2$ is normal in both Q_1 and Q_2. Sowe have both Q_1 and $Q_2 \subseteq N$. This implies $Q_1Q_2 \subseteq N$. But $|Q_1Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 27$, hence $|N| > 27$ (because 27 does not divide the order of$G$). But, we also see that $9 \mid |N|$. So, we have $|N| = 36, 72$ or 144, and thus $[G : N] \leq 4$, and by the cayley map, wehave $\phi : G \to \text{Perm}(G/N)$ to get $\ker \phi$ to be a nontrivial normal subgroup of G, hence G is not simple.

Example: Suppose $|G| = pqr$ with $p < q < r$ primes and suppose the sylow p-subgroup P is normal, and $G/P$is abelian. Prove that G is cyclic.
Proof: Let $\text{Syl}_q(G) = \{Q_1, \ldots, Q_n\}, n = n_q$ and let $H_i = Q_iP$. Then H_i is a subgroup of G, as $P \lhd G$. H_i/P is a subgroup of G/P, and thus $H_i/P \lhd G/P, |H_i| = pq, |H_i/P| = q$, and $|G/P| = qr$. So, H_i/P is asylow q-subgroup of G/P. But H_i/P is normal in G/P, soH_i/P is the only sylow subgroup of G/P, therefore $H_i/P = H_j/P \ \ \forall i, j$. This implies that $H_i = H_j := H$, and alsonote that $Q_i \subseteq H$ for all i. In fact, Q_i are sylow$q$-subgroups of H and since $p < q$, there is only one sylow$q$-subgroup of H, and hence $n_q = 1$. Similarly, we can arguethat $n_r = 1$. Thus, $P \lhd G, Q \lhd G, R \lhd G$, P, Q, Rcyclic of relatively prime orders, so by the lemma, $G = PQR$ iscyclic.