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Algebra 901 Notes
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Recall Sylow's second theorem:

1. If H is a p-subgroup of G, then $H \subseteq P$ for some $P \in \mathrm{Syl}_p(G)$.
2. $\mathrm{Syl}_p(G) = \{xPx^{-1} | x \in G\}$ for any $P \in \mathrm{Syl}_p(G)$. In other words, all Sylow p-subgroups are conjugate.
3. $\mathrm{n}_p = [G : \mathrm{N}_G(P)]$ for any $P \in \mathrm{Syl}_p(G)$.
4. $\mathrm{n}_p \equiv 1 \pmod{p}$.
5. $P \lhd G \Leftrightarrow \mathrm{n}_p = 1$.

Lemma: Let G be a group and H, K be cyclic subgroups of G of orders m and n respectively. Suppose that $H \lhd G$ and $K \lhd G$. Also suppose that $(m, n) = 1$. Then HK is cyclic of order mn.
Proof: Let $H = \langle x \rangle$ and $K = \langle y \rangle$. Note that $H \cap K = \{1\}$ since $(m, n) = 1$. Now consider $xyx^{-1}y^{-1}$. We know that it is in $H \cap K$ because $xyx^{-1} \in K$ since K is normal and $yx^{-1}y^{-1} \in H$ since H is normal. So, $xy = yx$. So, $\mathrm{o}(xy) = \mathrm{o}(x)\mathrm{o}(y)$, and $|HK| = \frac{|H||K|}{|H \cap K|} = mn$ so $HK = \langle xy \rangle$ and thus HK is cyclic.

Theorem: Suppose $|G| = pq$ where $p < q$ are primes, and $p \nmid (q - 1)$. Then G is cyclic.
Proof: Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. n_p divides $q = \frac{|G|}{p}$ and is congruent to 1 mod p. So, $\mathrm{n}_p \equiv 1$ or q, but $q \not\equiv 1 \pmod{p}$ by assumption. Therefore, we have $\mathrm{n}_p = 1$ which would imply that $P \lhd G$. Also, n_q divides p and $\mathrm{n}_q \equiv 1 \pmod{p}$ which implies $\mathrm{n}_q = 1$ which implies that $Q \lhd G$. So by the previous lemma, we have that PQ is cyclic of order pq, so $G = PQ$ is cyclic.

Example: Suppose $|G| = 6$. Then either $G \cong \mathbb{Z}_6$ or $G \cong \mathrm{S}_3$.
Proof: Let $P \in \mathrm{Syl}_2(G)$ and $Q \in \mathrm{Syl}_3(G)$. Then $Q \lhd G$ since $[G : Q] = 2$. So we have two cases, one where P is normal in G and one where P is not normal in G. If $P \lhd G$ then invoking the lemma we get that $G \cong \mathbb{Z}_6$. If $P \not\lhd G$ then considering that the Cayley map gives an isomorphism of groups, we see that $G \cong \mathrm{S}_3$.

Example: Let G be a group of order pqr where $p < q < r$ are primes. Then one of its Sylow subgroups is normal.
Proof: We know the following is true:

1. n_p divides $qr \quad \Rightarrow \quad \mathrm{n}_p = 1, q, r, qr$
2. n_q divides $pr \quad \Rightarrow \quad \mathrm{n}_q = 1, r, pr$
3. n_r divides $pq \quad \Rightarrow \quad \mathrm{n}_r = 1, pq$

Suppose that none of the Sylow subgroups are normal. Then $\mathrm{n}_p \geq q, \mathrm{n}_q \geq r, \mathrm{n}_r = pq$. Since $P_i \cap P_j = \{1\} \;\; \forall P_i, P_j \in \mathrm{Syl}_p(G)$, there are at least $q(p-1)$ non-identity elements in the sylow p subgroups. Similarly, there are at least $r(q-1)$ non-identity elements in the sylow q subgroups, and at least $pq(r-1)$

non-identity elements in the sylow r subgroups. So adding these up, we get $pqr + rq - q - r$ non-identity elements of G, and since $r > q \geq 2$, we get a contradiction.

Example: Let G be a group of order $2^3 \cdot 7 \cdot 11 = 616$. Prove the Sylow 11 subgroup is normal.
Proof: n_{11} divides $2^3 \cdot 7$ and $\mathrm{n}_{11} \equiv 1 \pmod{11}$, so $\mathrm{n}_{11} = 1$ or 56. Suppose $\mathrm{n}_{11} > 1$. Let $Q \in \mathrm{Syl}_{11}(G)$. So, we have $n_{11} = [G : \mathrm{N}_G(Q)] = 56$, but $[G : Q] = 56$ so by orders we have $Q = N_G(Q)$.
Claim: Either the Sylow 2-subgroup or the Sylow 7-subgroup of G is normal.
Proof: Suppose the sylow 7-subgroup is not normal. Then n_7 divides 88, and is congruent to 1 mod 7 which would imply that $\mathrm{n}_7 = 8$. There are 8(7-1) = 48 elements in the Sylow 7 subgroups and there are 56(11-1) = 560 elements in the sylow 11 subgroups, a total of 608. So, there are only enough elements left to form one Sylow 2 subgroup, and thus there is only one sylow 2 subgroup which would imply that the sylow 2 subgroup is normal.
So now, suppose the Sylow 2-subgroup is normal, and call it P. So, PQ is a subgroup of order 88. But Q is a sylow subgroup of PQ. By Sylow's second theorem, $Q \lhd PQ$ therefore $PQ \subseteq N_G(Q) = Q$, a contradiction. A similar argument works for the Sylow 7-subgroup as well. So, the Sylow 11 subgroup of G is normal.