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Algebra 901 Notes
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Sylow's Theorem I: If p^α divides the order of G with p a prime, then G has a subgroup of order p^α.

Corollary: Cauchy's Theorem: Suppose p divides the order of G. Then G has an element of order p.

Definition: Suppose p^n divides the order of G, but p^{n+1} does not. Then a subgroup of G of order p^n is called a Sylow p-subgroup of G.

Recall: In the following notes, let H, K be finite subgroups of a group G, and define $HK := \{hk | h \in H, k \in K\}$.

1. $\frac{|H||K|}{|H \cap K|} = |HK|$
2. HK is a subgroup of $G \Leftrightarrow HK = KH$
3. If H or K is normal, then HK is normal.
4. If $H \subseteq \mathrm{N}_G(K)$, then HK is a subgroup.

Definition: Suppose X is a G-set, $x \in X$ is called a fixed point of G if $G_x = G$, or equivalently, $Gx = \{x\}$.

Notation: Suppose p divides the order of G. Then $\mathrm{Syl}_p(G) := \{\text{all Sylow p-subgroups}\}$ and $\mathrm{n}_p := |\mathrm{Syl}_p(G)|$.

Remark: If H is a subgroup of order n and $x \in G$ then xHx^{-1} is also a subgroup of order n, and xHx^{-1} is called a conjugate of H. Therefore, any conjugate of a Sylow p-subgroup is a Sylow p-subgroup. In addition, the number of conjugates of a subgroup H is $[G : \mathrm{N}_G(H)]$.

Lemma 1: Let H be a p-subgroup of G and $P \in \mathrm{Syl}_p(G)$ and suppose $H \subseteq \mathrm{N}_G(H)$. Then $H \subseteq P$.
Proof: By one of the remarks, HP is a subgroup of G, and then

$$|HP| = \frac{|H||P|}{|H \cap P|} = |P| \cdot \left(\frac{|H|}{|H \cap P|} \right)$$
$$= |P| \cdot p^\alpha = p^{n+\alpha}$$

Therefore, $\alpha = 0$, since $P \subseteq HP$ but P is a sylow p-subgroup . So, this implies $H = H \cap P \subseteq P$.

Lemma 2: Let X be a G-set and suppose G is a p-group. Let n be the number of fixed points of G. Then $|X| \equiv n \pmod{p}$.
Proof:
$$|X| = \sum |Gx|$$

$$= n + \sum |Gx|$$

And since $|Gx|$ divides $|G|$, we know that

$$= n + \sum p^{\alpha_i}$$

For some $\alpha_i > 0$. Now modding out be p, we get

$$|X| \equiv n \pmod{p}.$$

Sylow's Second Theorem: Suppose for the following that p divides the order of G. Then

1. Any p-subgroup is contained in a Sylow p-subgroup.
2. All Sylow p-subgroups are conjugate.
3. $\mathrm{n}_p = [G : \mathrm{N}_G(P)]$ for any $P \in \mathrm{Syl}_p(G)$, and in particular n_p divides $|G|/|P| = \frac{|G|}{p^n}$.
4. $\mathrm{n}_p \equiv 1 \pmod{p}$.

Proof:

1. Let $P \in \mathrm{Syl}_p(G)$ and let $X = \{xPx^{-1} | x \in G\}$. Then $|X| = [G : \mathrm{N}_G(P)]$. Now let H be any p-subgroup of G, and let H act on X by conjugation (i.e. if $Q \in X$ then $h \cdot Q = hQh^{-1}$. Note that $p \nmid |X|$, since $P \subseteq \mathrm{N}_G(X)$. By lemma 2, we have $|X| \equiv$ fixed points of $H \pmod{p}$. Since $|X| \not\equiv 0$ (mod p, $\exists$ a fixed point of H in X, say Q. So, we know $hQh^{-1} = Q \; \forall \; h \in H$. But $Q \in X \subseteq \mathrm{Syl}_p(G)$. The above implies that $H \subseteq \mathrm{N}_G(Q)$ and by lemma 1, we have $H \subseteq Q$.
2. Let $P' \in Syl_p(G)$. By replacing H by P' in the previous argument, we get $P' \subseteq Q = xPx^{-1}$(since xPx^{-1} has the same order of Q). Hence any two Sylow p-subgroups are conjugates, and thus $X = \mathrm{Syl}_p(G)$.
3. By definition, $\mathrm{n}_p := |X| = [G : \mathrm{N}_G(P)]$.
4. Choose $P \in Syl_p(G)$ and $X = \{xPx^{-1} \mid x \in G\} = \mathrm{Syl}_p(G)$. Let P act on X by conjugation. By the same argument as in 1, there exists a fixed point of P in X, say Q. This means that $yQy^{-1} = Q \; \forall \; y \in P$, thus $P \subseteq \mathrm{N}_G(Q)$ which implies $P \subseteq Q$ which also implies $P = Q$. Hence, there is only one fixed point of P in X. By lemma 2, $\mathrm{n}_p = |X| \equiv$ number of fixed points $\equiv 1 \pmod{p}$.

Corollary: Let $P \in \mathrm{Syl}_p(G)$. Then $P \lhd G \Leftrightarrow P$ is the unique sylow p-subgroup of G.