Recall the quadratic formula: The roots of \( f(x) = ax^2 + bx + c \in K[x] \) are 
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
if the characteristic of \( K \) is not 2, and where \( \sqrt{b^2 - 4ac} \) is a root of \( x^2 - (b^2 - 4ac) \). Thus, the roots of \( f(x) \) lie in the field \( K(\alpha) \) where \( \alpha \) is a root of \( x^2 - (b^2 - 4ac) \).

Also, note that Cardano in the 1500s found a formula to solve the general cubic. Indeed, let \( f(x) = x^3 + ax^2 + bx + c \in K[x] \). Assume that \( \text{Char } K \neq 2, 3 \). Replacing \( x \) by \( x - \frac{1}{3}a \), we may eliminate the \( x^2 \) term, so we may assume that \( f(x) = x^3 + px + q \). Cardano’s formula shows that the roots of \( f(x) \) lie in the field \( K(\omega, \delta, y_1, y_2) \) where \( \omega \) is a primitive \( n \)th root of unity, \( \delta = \sqrt{12pq + 81q^2} \) and \( y_1, y_2 = \sqrt{-\frac{27}{4}}q \pm \frac{3}{2} \delta \).

So we have a tower of fields (called a root tower):

\[
\begin{align*}
E &= K(\omega, \delta, y_1, y_2) = F_3(y_2) \\
F_3 &= F_2(y_1) \\
F_2 &= F_1(\delta) \\
F_1 &= K(\omega) \\
F_0 &= K
\end{align*}
\]

**Definition:** A finite extension \( E/K \) is called a **radical extension** if \( \exists u_1, \ldots, u_n \) such that for each \( i \) \( \exists u_i \in K(u_1, \ldots, u_{i-1}) \) and \( E = K(u_1, \ldots, u_n) \). The sequence

\[
K \subseteq K(u_1) \subseteq K(u_1, u_2) \subseteq \cdots \subseteq K(u_1, \ldots, u_n) = E
\]

is called a **root tower**. A polynomial in \( K[x] \) is said to be **Solvable by Radicals** if \( f(x) \) splits in some radical extension.

**Theorem:** Let \( f(x) \in K[x] \) be a separable polynomial and \( E \) a splitting field for \( f(x) \). Assume that \( \text{Char } K \nmid [E : K] \). If \( \text{Gal}_K(f(x)) \) is solvable, then \( f(x) \) is solvable by radicals.

**Proof:** Let \( n = [E : K] \) and \( \omega \) a primitive \( n \)th root of unity. Consider the diagram
By a homework exercise, $E(\omega)/K(\omega)$ is Galois. Furthermore, we have that $\text{Gal}(E(\omega)/K(\omega)) \cong$ a subgroup of $\text{Gal}(E/K)$. As subgroups of solvable groups are solvable, $\text{Gal}(E(\omega)/K(\omega))$ is solvable. If we show that $E(\omega)/K(\omega)$ is a radical extension, then certainly $E(\omega)/K$ is a radical extension, since adjoining $\omega$ is a radical extension. Of course $f$ splits in $E(\omega)$ as it splits in $E$. Hence without loss of generality, we may replace $K$ with $K(\omega)$ and assume $K$ contains all the $n$th roots of unity. Also, $[E(\omega) : K(\omega)] = |\text{Gal}(E(\omega)/K(\omega))|$ which divides $|\text{Gal}(E/K)|$ by Lagrange. Hence $\text{Char } k \nmid [E(\omega) : K(\omega)]$. So, let $G = \text{Gal}(E/K)$. As $G$ is solvable, there is a normal series

$$\{1\}G_t \triangleleft G_{t-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

with $G_t/G_{t+1} \cong C_{p_t}$ (the cyclic group on $p_t$ elements). We may assume that we get cyclic groups since any refinement of a solvable series is solvable, and any composition series (i.e. a normal series that all the factor groups are simple) for a solvable group has factor groups that are cyclic of prime order. So we have the correspondence

<table>
<thead>
<tr>
<th>Group Side</th>
<th>Field Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1}$</td>
<td>$E = F_t$</td>
</tr>
<tr>
<td>$G_{k-1}$</td>
<td>$F_{t-1}$</td>
</tr>
<tr>
<td>$G_{k-2}$</td>
<td>$F_{t-2}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$G_1$</td>
<td>$F_1$</td>
</tr>
<tr>
<td>$G_0$</td>
<td>$F_0 = K$</td>
</tr>
</tbody>
</table>

2
Where the extensions are normal (and hence Galois) because the series of groups on the left are normal. So, $F_i/F_{i-1}$ is Galois and $\text{Gal}(F_i/F_{i-1}) \cong G_i/G_{i-1} \cong C_{p_i}$ and $K$ contains primitive $p_i$th roots of unity, $\omega^{n/p_i}$. Note that $p_i \nmid [E : K]$ so $\text{Char} K \nmid p_i$. By the theorem on cyclic extensions, we have that $F_i = F_{i-1}(u)$ where $u^{p_i} \in F_{i-1}$. Hence the above is a root tower, and hence $E/F$ is solvable by radicals.