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Algebra 901 Notes
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Definition: Let G be a group, and H a subgroup of G. Then H is called a characteristic subgroup of G provided $\phi(H) = H$ for all $\phi \in \text{Aut}(G)$. We will write H Char G. Note that in general, it is enough to show that $\phi(H) \subseteq H$ for all $\phi \in \text{Aut}(G)$.

Remarks:

1. H Char G implies that $H \lhd G$, since $\text{Inn}(G) \subset$(indeed a normal subgroup of) $\text{Aut}(G)$.
2. $H \lhd G \not\Rightarrow H$ Char G. For example, let $G = V_4$(the Klein 4-group). Then $\phi(\langle a \rangle) = \langle b \rangle$ for an automorphism ϕ, but $\langle a \rangle$ is normal in G. Therefore, $\langle a \rangle$ is not characteristic.
3. If $|H| < \infty$ and H is the unique subgroup of order $|H|$, then H Char G, just by comparing the orders of H and $\phi(H)$ and noting that since ϕ is an automorphism we have that $\phi(H) = H$ because H is the only subgroup of that order.
4. If G is cyclic of finite order, then every subgroup is characteristic. (Corollary of the above)
5. If $P \in \text{Syl}_p(G)$ then $P \lhd G \Leftrightarrow P$ Char G. (Also a corollary to the above)
6. If $K \lhd H$ and $H \lhd G$ we know that this does not imply that $K \lhd G$. (Take, for example, D_8, and $H = \{1, x^2, y, x^2y\}$ and $K = \langle y \rangle$. Then $K \lhd H$ and $H \lhd G$ but $K \not\lhd G$.
7. We do have the following facts, however:
    (a) K Char H and H Char G implies that K Char G. (restrict an automorphism of G to H to get an automorphism of H since H Char G. Then apply this to K).
    (b) K Char H and $H \lhd G$ implies that $K \lhd G$. (consider inner automorphisms of G)
    (c) Let $P \in \text{Syl}_p(G)$. Then $P \lhd H$ and $H \lhd G$ implies that $P \lhd G$.

Definition: Let G be a group. Then the commutator subgroup of G is the subgroup generated by the set $\{aba^{-1}b^{-1} \mid a, b \in G\}$, and is denoted by G' or $G^{(1)}$.

Remarks:

1. G is abelian if and only if $G' = \{1\}$.
2. G' Char G : Let $\phi \in \text{Aut}(G)$. It's enough to show that $\phi(G') \subseteq G$. Let $aba^{-1}b^{-1} \in G'$. Then $\phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} \in G'$, hence G' Char G. Hence in particular, we have that $G \lhd G$.

Proposition: Let H be a subgroup of G. Then $G' \subseteq H \Leftrightarrow H \lhd G$ and G/H is abelian.
Proof: ”$\Rightarrow$”: G' Char G so $G' \lhd G$, but in G/G', we have that $xy = yx$ so that G/G' is abelian. Therefore, $H/G' \lhd G/G'$ by the correspondence theorem, and therefore, $H \lhd G$(?). Also, the third isomprphism theorem tells you that G/H is the quotient of abelian groups and hence abelian.

”$\Leftarrow$”: Suppose $H \lhd G$ and G/H is abelian. Let $a, b \in G$. Then $aHbH = bHaH$ which implies that $aba^{-1}b^{-1} \in H$, so $G' \subseteq H$.

Definition: Let higher commutator subgroups of G we define as follows: $G^{(0)} = G$, $G^{(1)} = G'$, and in general, $G^{(i+1)} = (G^{(i)})'$ (the commutator subgroup of the ith commutator). The sequence of subgroups

$$\cdots \lhd G^{(i+1)} \lhd G^{(i)} \lhd \cdots \lhd G^{(1)} \lhd G^{(0)} = G$$

is called the **derived series** of G. Note that any factor groups of the above are abelian, and furtherfore we have that $G^{(i+1)}$ Char $G^{(i)}$ and by transitivity we have that $G^{(i)}$ Char G for all i.

Definition: A sequence of subgroups of G

$$\{1\} = H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_0 = H$$

is called a **normal series** for G. The factor groups for the series are $\{H_i/H_{i+1}\}$.
A **solvable series** for G is a normal series in which the factor groups are abelian. G is said to be **solvable** if it has a solvable series.

Examples:

1. If G is abelian, then G is solvable.
2. Let $G = D_{2n} = C_n \rtimes C_2$. Then $\{1\} \lhd C_n \lhd D_{2n}$ is a solvable series for D_{2n}.
3. Let $G = A_4$. We know that $P \lhd A_4$ where $P \in \mathrm{Syl}_2(G)$. Therefore, we have that $\{1\} \lhd P \lhd A_4$ is a solvable series for A_4 (in fact, $A_4' = P$).

Theorem: Let G be a group. Then G is solvable $\Leftrightarrow$ $G^{(n)} = \{1\}$ for some n.
Proof: ”$\Leftarrow$” This direction is trivial since if $G^{(n)} = \{1\}$ for some n, then the derived series for G gives us a solvable series.
”$\Rightarrow$”: Suppose that $\{1\} = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$ is a solvable series for G. First we prove a claim:

Claim: $G^{(i)} \subseteq G_i$ for all i.
Proof of Claim: The case $i = 0$ is trivial, since both are G. So, suppose that $G^{(i)} \subseteq G_i$ for some i. We know that G_i/G_{i+1} is abelian, so $G_{i+1} \supset G_i' \supset (G^{(i)})' = G^{(i+1)}$. Therefore, $G^{(n)} = G_n = \{1\}$. Therefore, we have that $G^{(n)} = \{1\}$, as desired. The gist of this argument is that the derived series is as short as any solvable series could possibly be, however it is in practice often very hard to compute all the derived subgroups by hand.

Lemma: Let $\phi : A \to B$ be a group homomorphism. Then $\phi(A^{\{}(n)\}) = \phi(A)^{(n)} = (\mathrm{im}\phi)^{\{}(n)\}$, for all $n \geq 0$.
Proof: We proceed by induction on n. When $n = 0$, the result is trivial. Let $n = 1$. Since A' is generated by $S = \{aba^{-1}b^{-1}\}$ we know that $\phi(A')$ is generated by $\phi(S) = \{\phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1}\}$. But $\phi(A)'$ is generated by the same elements, so $\phi(A)' = \phi(A')$. The general case is identical to this one.

Corollary: Let $\phi : A \to B$ be a group homomorphism. If A is solvable then so is $\mathrm{im}\phi$.

Theorem: Let G be a group, and H a subgroup of G. Then if G is solvable then so is H. Furthermore, suppose $H \lhd G$. Then G is solvable $\Leftrightarrow$ H is solvable and G/H is solvable.

Proof: Clearly, we have that $H^{(i)} \subseteq G^{(i)}$. If G is solvable then $G^{(n)} = \{1\}$ for some n and hence $H^{(n)} = \{1\}$, so H is solvable. For the second assertion, let us prove the forward direction first. So, let G be solvable. By the first assertion, H is solvable. Consider $\phi : G \to G/H$ the natural surjection. By the corollary, we have that G/H is solvable. For the reverse direction, suppose H and G/H are solvable. Let $\phi : G \to G/H$ be as above. We know G/H is solvable so $(G/H)^{(n)} = \{1\}$. But $\phi(G^{(n)}) = \phi(G)^{(n)} = (G/H)^{(n)} = \{1\}$. Therefore, $G^{(n)} \subset \ker \phi = H$. As H is solvable, $H^{(k)} = \{1\}$ for some k. So $G^{(n+k)} = \{1\}$ so that G is solvable.

Theorem: Let G be a p-group, p a prime. Then G is solvable.
Proof: Say $|G| = p^n$. Use induction on n. If $n \leq 2$, then G is abelian so automatically solvable. Suppose that $n \geq 3$. We know that $\exists H \leq G$ so that $|H| = p^{n-1}$. Since $[G : H] = p$ we immediately have that $H \lhd G$. Thus H is solvable by induction, and G/H is cyclic so solvable. So G is solvable.

Example: Any group of order $1998 = 2 \cdot 3^3 \cdot 37$ is solvable.
Proof: Let $Q \in \mathrm{Syl}_37(G)$. By Sylow's thoerems, $Q \lhd G$, and also Q is cyclic, so Q is solvable. So it is enough to show that G/Q is solvable. Let G' be any group of order $2 \cdot 3^3$. Then the sylow 3-subgroup is normal and by the last theorem, it is solvable. Also, G'/P is cyclic, so solvable, thus G' is solvable.

Burnside's Theorem (1904): Any group of order $p^n q^l$ where p and q are primes, is solvable.

Feit-Thompson's Theorem (1963): Any group of odd order is solvable.

Theorem: A_5 is simple. First we prove a couple of claims:

Claim 1: Suppose that $H \lhd A_5$, where H is a proper subgroup. Then $5 \nmid |H|$.
Proof of Claim 1: Suppose $5 \mid |H|$. Then H contains a sylow 5-subgroup of A_5. As $H \lhd A_5$, H contains all conjugates of this Sylow 5-subgroup. Therefore H contains all elements of order 5. The elements of order 5 are 5-cycles. There are 4! 5-cycles. So H contains at least 24 elements plus the identity so 25 elements. Since $|H| \mid 60$ and $|H| < 60$, $|H| = 30$. Buy any group of order 30 has a normal Sylow 30-subgroup by our earlier discussion, hence a contradiction.

Claim 2: If A_5 is not simple then A_5 contains a normal subgroup of order 2,3, or 4.
Proof of Claim 2: Let H be a proper normal subgroup of A_5. By Claim 1, $|H| = 2, 3, 4, 6$ or 12. If $|H| = 6$ then then Sylow 3-subgroup of H (and of A_5) is normal by Sylow's theorems. Therefore $P \lhd G$. If $|H| = 12$, then the Sylow 2 or Sylow 3 subgroup of H is normal in H and hence in A_5.

Proof of Theorem: Assume that A_5 is not simple. Let K be the normal subgroup given by claim 2. Then $|A/K| = 30, 20, 15$. Then the sylow 5-subgroup H/K or A/K is normal. Then $H \lhd A_5$, but $|H/K| = 5$ contradicting claim 1. Therefore A_5 is simple. Note that A_n is simple for $n \geq 5$.

Corollary: S_n is not solvable for $n \geq 5$. Indeed, if S_5 were solvable, then so is A_5. But A_5 is simple and nonabelian, so not solvable. Hence S_5 is not solvable. Since S_5 is isomorphic to a subgroup of S_n for $n \geq 5$, S_n is not solvable.