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**Algebra 901 Notes**
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**Major Example II:** Classify all groups up to order $20 = 2^2 \cdot 5$. Let $P \in \mathrm{Syl}_2(G)$ and $Q \in \mathrm{Syl}_5(G)$. By Sylow's Thoerems, we immediately have that $Q \lhd G$. Also, set $Q = \langle y \rangle$. Therefore, we have that $G \cong Q \rtimes_\phi P$, where $\phi : P \to \mathrm{Aut}(Q) \cong \mathbb{Z}_5^*$.

1. Case 1: $P \cong C_4$. Let $P = \langle x \rangle$. Then we know that $\phi(x) =$an automorphism of order 1,2,or 4.
   (a) If $o(\phi(x)) = 1$, then $\phi$ is the trivial map and so $G = Q \times P \cong C_{20}$.
   (b) If $o(\phi(x)) = 2$, then $\phi$ is the inversion map, so we have that $G = \langle x, y \mid x^4 = 1, y^5 = 1, xyx^{-1} = y^4 \rangle$.
   (c) If $o(\phi(x)) = 4$, then in this cae $\phi(P) = \mathrm{Aut}(Q)$. By the theorem from last class any two such *phi*'s give rise to isomorphic semidirect products. Let $\phi(x)$ be the automorphism sending $y \mapsto y^2$. Then $xyx^{-1} = y^2$ so we have that $G = \langle x, y \mid x^4, y^5, xyx^{-1} = y^2 \rangle$.
2. Case 2: $P \cong C_2 \times C_2$. Consider $\ker \phi$ where $\phi : P \to \mathrm{Aut}(Q)$. Note that $|\ker \phi| = 1, 2 or 4$.
   (a) $|\ker \phi| = 4$: Then $\phi$ is the trivial map and hence $G = C_2 \times C_2 \times C_5$.
   (b) $|\ker \phi| = 1$: Cannot happen since $C_2 \times C_2 \not\cong \mathbb{Z}_5^*$.
   (c) $|\ker \phi| = 2$: Let $x \in \ker \phi \setminus \{1\}$. Therefore $\phi(x)$, which is conjugation by $x$, induces the trivial map on $Q$, so that $xqx^{-1} = q$ for all $q \in Q$. Let $H = \langle x \rangle Q$. Then $|H| = 10$ and $H$ is abelian so we have that $H \cong C_{10}$. Also, by indexes, $H$ is normal in $G$. So, glue the groups together via the inversion map again and you get $D_{20}$ as in the case where we were looking at groups of order 12.

**Lemma:** Suppose $m \nmid n$ where $m, n \in \mathbb{Z}^+$. Let $s \in \mathbb{Z}$ such that $(s, m) = 1$. Then there exists a $t \in \mathbb{Z}$ such that $(s + tm, n) = 1$.

**Lemma:** Let $\phi : C_n \to C_m$ be a surjective group homomorphism, and let $C_n = \langle a \rangle$ and $C_m = \langle b \rangle$. Then $b = \phi(a^r)$ where $(r, n) = 1$. Just use the above lemma to prove this result - follow your nose.

**Theorem:** Let $K$ be a cyclic group of order $n$ and $H$ an arbitrary group. Let $\phi_1, \phi_2 : K \to \mathrm{Aut}(H)$ be two group homomorphisms such that $\phi_1(K)$ and $\phi_2(K)$ are conjugate. Then $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.
*Proof:* Let $K = \langle a \rangle \cong C_n$. Then $\phi_2(K) = \sigma\phi_1(K)\sigma^{-1}$ for some $\sigma \in \mathrm{Aut}(H)$. Then note that $\sigma\phi_1(K)\sigma^{-1} = \sigma\phi_1(\langle a \rangle)\sigma^{-1} = \langle \sigma\phi_1(a)\sigma^{-1} \rangle$. Now apply the lemma to $\phi : K \to \langle \sigma\phi_1(a)\sigma^{-1} \rangle$. Then $\sigma\phi_1(a)\sigma^{-1} = \phi_2(a^r)$ for some $r$ with $(r, n) = 1$. Now, notice that for any $s \in \mathbb{Z}$, we have that

$$\sigma\phi_1(a^s)\sigma^{-1} = \sigma\phi_1(a)^s\sigma^{-1} = (\sigma\phi_1(a)\sigma^{-1})^s = \phi_2(a^r)^s = \phi_2((a^s)^r)$$

Therefore, for any $x \in K$, we have that $\sigma\phi_1(x)\sigma^{-1} = \phi_2(x^r)$, i.e. $\sigma\phi_1(x) = \phi_2(x^r)\sigma$. Now we may define our isomorphism:

$$f : H \rtimes_{\phi_1} K \to H \rtimes_{\phi_2} K$$
$$(h, k) \mapsto (\sigma(h), k^r)$$

Now we check that it is in fact an isomorphism.

- **1-1**: If $f(h,k) = (1,1)$, then $\sigma(h) = 1$ and so $h = 1$ since $\sigma$ was an automorphism of $H$. If $k^r = 1$, $o(k) \mid r$ so $o(k) = 1$ since we had that $(r,n) = 1$. Therefore, $k = 1$.
- **Onto:** Let $(h', k') \in H \rtimes_{\phi_2} K$. Since $\sigma$ is onto, $\exists h \in H$ such that $\sigma(h) = h'$. Since $(r,n) = 1$, $rs + nt = 1$ for suitable $s$ and $t$. Then $k' = (k')^1 = (k')^{rs+nt} = ((k')^s)^r$ (as the $n$ term goes away since $K$ is cyclic of order $n$). So, let $k = (k')^s$. Then $f(h,k) = (h', k')$.
- **Hom:**

$$\begin{aligned} f((h_1,k_1)(h_2,k_2)) &= f(h_1\phi_1(k_1)h_2, k_1k_2) \\ &= (\sigma(h_1\phi_1(k_1)(h_2)), k_1^r k_2^r) \\ &= (\sigma(h_1)\sigma(\phi_1(k_1)(h_2)), k_1^r k_2^r) \\ &= (\sigma(h_1)\phi_2(k_1^r)\sigma(h_2), k_1^r k_2^r) \\ &= (\sigma(h_1), k_1^r)(\sigma(h_2), k_2^r) \\ &= f(h_1,k_1)f(h_2,k_2) \end{aligned}$$

**Major Example III:** Let's compute all groups of order 30. Let $P \in \mathrm{Syl}_2(G)$, $Q \in \mathrm{Syl}_3(G)$ and $R \in \mathrm{Syl}_5(G)$.

**Either $Q$ or $R$ are normal:**
*Proof:* If not, $n_3 = 10$ and $n_5 = 6$ by Sylow's theorems. This is way too many elements since the intersections of these have to be trivial.

So, let $P = \langle x \rangle$. By the above claim, we have that $QR$ is a subgroup of $G$, and we know that $QR$ is the cyclic group of order 15. Furthermore, since $[G : QR] = 2$, we know that $QR \lhd G$. Hence, $G \cong QR \rtimes_\phi P$ where $\phi : P \to \mathrm{Aut}(QR) \cong \mathbb{Z}_{15}^*$. If $\phi$ is trivial, then $\phi$ is cyclic of order 30, and if $\phi$ is the map that sends $x$ to inversion, then $G = D_{30}$ as before. So, we have that $G \cong C_{15} \rtimes_\phi C_2$ where $\phi : C_2 \to \mathrm{Aut}(C_{15}) \cong Z_{15}^*$. Note that there are 3 elements of order 2 in $\mathrm{Aut}(C_{15})$, namely $-1$(inversion), 4 and 11. So we have 3 nontrivial choices for $\phi(a)$:

1. $\phi(a) : C_{15} \to C_{15}$ where $\phi(a)(b) = b^{-1}$, the inversion map. Here, $G \cong D_{30}$.
2. $\phi(a) : C_{15} \to C_{15}$ where $\phi(a)(b) = b^4$. Then we have a presentation $\langle x, y \mid x^2 = 1, y^{15} = 1, xyx^{-1} = y^4 \rangle$.
3. $\phi(a) : C_{15} \to C_{15}$ where $\phi(a)(b) = b^{11}$. Then we have a presentation $\langle x, y \mid x^2 = 1, y^{15} = 1, xyx^{-1} = y^{11} \rangle$.

Now how can we tell these groups apart? Lets consider their centers. The possible orders for $Z(G)$ to force $G$ to be a nonabelian group are $1, 3, 5$ (since if $G/Z(G)$ is cyclic, $G$ is abelian). In all cases, note that $Z(G)$ is cyclic, so lets say that $Z(G) = \{1\}$ or $Z(G) = \langle b^5 \rangle$, the unique Sylow 3-subgroup or $Z(G) = \langle b^3 \rangle$, the unique Sylow 5-subgroup. It can be checked that each of the above groups has exactly one of the above as its center. Just look at $\phi(a)(b^5)$ and $\phi(a)(b^3)$ to see if they are in the kernel, as that means conjugation by that element is trivial, i.e. it is in the center.