Major Example II: Classify all groups up to order $20 = 2^2 \cdot 5$. Let $P \in \text{Syl}_2(G)$ and $Q \in \text{Syl}_5(G)$. By Sylow’s Theorems, we immediately have that $Q \triangleleft G$. Also, set $Q = \langle y \rangle$. Therefore, we have that $G \cong Q \rtimes P$, where $\phi : P \to \text{Aut}(Q) \cong \mathbb{Z}_5^\times$.

1. Case 1: $P \cong C_4$. Let $P = \langle x \rangle$. Then we know that $\phi(x)$ is an automorphism of order 1, 2, or 4.
   (a) If $o(\phi(x)) = 1$, then $\phi$ is the trivial map and so $G = Q \times P \cong C_{20}$.
   (b) If $o(\phi(x)) = 2$, then $\phi$ is the inversion map, so we have that $G = \langle x, y \mid x^4 = 1, y^5 = 1, xyx^{-1} = y^4 \rangle$.
   (c) If $o(\phi(x)) = 4$, then in this case $\phi(P) = \text{Aut}(Q)$. By the theorem from last class any two such $phi's$ give rise to isomorphic semidirect products. Let $\phi(x)$ be the automorphism sending $y \mapsto y^2$. Then $xyx^{-1} = y^2$ so we have that $G = \langle x, y \mid x^4, y^5, xyx^{-1} = y^2 \rangle$.

2. Case 2: $P \cong C_2 \times C_2$. Consider $\ker \phi$ where $\phi : P \to \text{Aut}(Q)$. Note that $|\ker \phi| = 1, 2 \text{ or } 4$.
   (a) $|\ker \phi| = 4$: Then $\phi$ is the trivial map and hence $G = C_2 \times C_2 \times C_5$.
   (b) $|\ker \phi| = 1$: Cannot happen since $C_2 \times C_2 \not\cong \mathbb{Z}_5^*$. 
   (c) $|\ker \phi| = 2$: Let $x \in \ker \phi \setminus \{1\}$. Therefore $\phi(x)$, which is conjugation by $x$, induces the trivial map on $Q$, so that $xqx^{-1} = q$ for all $q \in Q$. Let $H = \langle x \rangle Q$. Then $|H| = 10$ and $H$ is abelian so we have that $H \cong C_{10}$. Also, by indexes, $H$ is normal in $G$. So, glue the groups together via the inversion map again and you get $G_{20}$ as in the case where we were looking at groups of order 12.

Lemma: Suppose $m \nmid n$ where $m, n \in \mathbb{Z}^+$. Let $s \in \mathbb{Z}$ such that $(s, m) = 1$. Then there exists a $t \in \mathbb{Z}$ such that $(s + tn, n) = 1$.

Lemma: Let $\phi : C_n \to C_m$ be a surjective group homomorphism, and let $C_n = \langle a \rangle$ and $C_m = \langle b \rangle$. Then $b = \phi(a^r)$ where $(r, n) = 1$. Just use the above lemma to prove this result - follow your nose.

Theorem: Let $K$ be a cyclic group of order $n$ and $H$ an arbitrary group. Let $\phi_1, \phi_2 : K \to \text{Aut}(H)$ be two group homomorphisms such that $\phi_1(K)$ and $\phi_2(K)$ are conjugate. Then $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.

Proof: Let $K = \langle a \rangle \cong C_n$. Then $\phi_2(K) = \sigma \phi_1(K) \sigma^{-1}$ for some $\sigma \in \text{Aut}(H)$. Then note that $\sigma \phi_1(K)\sigma^{-1} = \sigma \phi_1(\langle a \rangle)\sigma^{-1} = \langle \sigma \phi_1(a)\sigma^{-1} \rangle$. Now apply the lemma to $\phi : K \to \langle \sigma \phi_1(a)\sigma^{-1} \rangle$. Then $\phi \phi_1(a)\sigma^{-1} = \phi_2(a^r)$ for some $r$ with $(r, n) = 1$. Now, notice that for any $s \in \mathbb{Z}$, we have that $\sigma \phi_1(a^s)\sigma^{-1} = \sigma \phi_1(a^s)\sigma^{-1} = \langle \sigma \phi_1(a)\sigma^{-1} \rangle^s = \phi_2(a^s) = \phi_2((a^r)^s)$.

Therefore, for any $x \in K$, we have that $\sigma \phi_1(x)\sigma^{-1} = \phi_2(x^r)$, i.e. $\sigma \phi_1(x) = \phi_2(x^r)\sigma$. Now we may define our isomorphism:

$$f : H \rtimes_{\phi_1} K \to H \rtimes_{\phi_2} K$$

$$(h, k) \mapsto (\sigma(h), k^r)$$

Now we check that it is in fact an isomorphism.
• **1-1:** If \( f(h, k) = (1, 1) \), then \( \sigma(h) = 1 \) and so \( h = 1 \) since \( \sigma \) was an automorphism of \( H \). If \( k' = 1 \), \( o(k) \mid r \) so \( o(k) = 1 \) since we had that \( (r, n) = 1 \). Therefore, \( k = 1 \).

• **Onto:** Let \( (h', k') \in H \times_{\phi_2} K \). Since \( \sigma \) is onto, \( \exists h \in H \) such that \( \sigma(h) = h' \). Since \( (r, n) = 1 \), \( rs + nt = 1 \) for suitable \( s \) and \( t \). Then \( k' = (k')^1 = (k')^{rs + nt} = ((k')^r)^s \) (as the \( n \) term goes away since \( K \) is cyclic of order \( n \)). So, let \( k = (k')^s \). Then \( f(h, k) = (h', k') \).

• **Hom:**

\[
\begin{align*}
\phi((h_1, k_1)(h_2, k_2)) &= \phi(h_1 \phi_1(k_1)h_2, k_1k_2) \\
&= (\sigma(h_1)\phi_1(k_1)h_2, k_1^r k_2^s) \\
&= (\sigma(h_1)\sigma_2(k_1^r)h_2, k_1^r k_2^s) \\
&= (\sigma(h_1), k_1^r)(\sigma(h_2), k_2^s) \\
&= f(h_1, k_1)f(h_2, k_2)
\end{align*}
\]

**Major Example III:** Let’s compute all groups of order 30. Let \( P \in \text{Syl}_2(G) \), \( Q \in \text{Syl}_5(G) \) and \( R \in \text{Syl}_3(G) \).

**Either \( Q \) or \( R \) are normal:**

*Proof:* If not, \( n_3 = 10 \) and \( n_5 = 6 \) by Sylow’s theorems. This is way too many elements since the intersections of these have to be trivial.

So, let \( P = \langle x \rangle \). By the above claim, we have that \( QR \) is a subgroup of \( G \), and we know that \( QR \) is the cyclic group of order 15. Furthermore, since \( [G : QR] = 2 \), we know that \( QR \triangleleft G \). Hence, \( G \cong QR \rtimes_\phi P \) where \( \phi : P \to \text{Aut}(QR) \cong \mathbb{Z}_{15}^* \). If \( \phi \) is trivial, then \( \phi \) is cyclic of order 30, and if \( \phi \) is the map that sends \( x \) to inversion, then \( G = D_{30} \) as before. So, we have that \( G \cong C_{15} \rtimes_\phi C_2 \) where \( \phi : C_2 \to \text{Aut}(C_{15}) \cong \mathbb{Z}_{15}^* \).

Note that there are 3 elements of order 2 in \( \text{Aut}(C_{15}) \), namely \(-1(\text{inversion}), 4 \) and \( 11 \). So we have 3 nontrivial choices for \( \phi(a) \):

1. \( \phi(a) : C_{15} \to C_{15} \) where \( \phi(a)(b) = b^{-1} \), the inversion map. Here, \( G \cong D_{30} \).
2. \( \phi(a) : C_{15} \to C_{15} \) where \( \phi(a)(b) = b^4 \). Then we have a presentation \( \langle x, y \mid x^2 = 1, y^{15} = 1, xyx^{-1} = y^4 \rangle \).
3. \( \phi(a) : C_{15} \to C_{15} \) where \( \phi(a)(b) = b^{11} \). Then we have a presentation \( \langle x, y \mid x^2 = 1, y^{15} = 1, xyx^{-1} = y^{11} \rangle \).

Now how can we tell these groups apart? Let’s consider their centers. The possible orders for \( Z(G) \) to force \( G \) to be a nonabelian group are 1, 3, 5 (since if \( G/Z(G) \) is cyclic, \( G \) is abelian). In all cases, note that \( Z(G) \) is cyclic, so let’s say that \( Z(G) = \{1\} \) or \( Z(G) = \langle b^5 \rangle \), the unique Sylow 3-subgroup or \( Z(G) = \langle b^3 \rangle \), the unique Sylow 5-subgroup. It can be checked that each of the above groups has exactly one of the above as its center. Just look at \( \phi(a)(b^5) \) and \( \phi(a)(b^3) \) to see if they are in the kernel, as that means conjugation by that element is trivial, i.e. it is in the center.