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Algebra 901 Notes
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Group Actions on Sets

Definition : Let G be a group on X a set. A **group action** (or just action) of G on X is a mapping:

$$G \times X \to X$$
$$(g, x) \mapsto gx$$

Such the following properties hold:

$$g_1(g_2x) = (g_1g_2)x \qquad \forall g_1, g_2 \in G$$
$$1x = x \qquad \forall x \in X$$

Examples:

1. Translation of cosets : Let H be a subgroup of a group G, and let G/H denote the set of left cosets of H (Think $X = G/H$).

   Let G act on G/H by $g \cdot (g'H) := (gg')H \ \ \forall \ \ g, g' \in G$. The above defines an action (Check this out). (By definition, property (i) holds, as well as property (ii).

2. Conjugation : Let G act on itself via $g \cdot x := gxg^{-1} \ \ \forall \ g, x \in G$. (Also check this out.)(we have the following:)

$$\begin{aligned} g_1 \cdot (g_2 \cdot x) &= g_1(g_2xg_2^{-1})g_1^{-1} \\ &= g_1g_2x(g_2^{-1}g_1^{-1}) \\ &= g_1g_2x(g_1g_2)^{-1} \\ &= (g_1g_2) \cdot x \end{aligned}$$

Remarks:

1. Suppose G acts on X. X is then called a G-set. Further suppose that $g \in G$. Then the map:

$$\phi_g : X \to X \quad \text{is bijective}$$
$$x \mapsto gx$$

2. So, we have that ϕ_g is just a permutation of X, and we write $\phi_g \in \mathrm{Perm}(X)$. Thus, each element of G gives rise to a permutation of X. So this gives rise to a homomorphism:

$$\phi : G \to \mathrm{Perm}(X)$$
$$g \mapsto \phi_g$$

Check that this is a group homomorphism. We have that $\phi_{gg'} : X \to X$ sending $x \mapsto gg'x$. So clearly we have $\phi(gg') = \phi_{gg'}$ by definition. We also have the following:

$$\phi(g)\phi(g') = \phi_g \phi_{g'}$$

which sends $x \overset{\phi_{g'}}{\mapsto} g'x \overset{\phi_g}{\mapsto} gg'x$. So the group structure is indeed preserved.

Definition: The kernel of the above mapping is called the **kernel of the action**.

$$g \in \text{the kernel of the action} \Leftrightarrow gx = x \quad \forall x \in X$$

Examples:

1. In the translation of cosets (G acting on G/H) the kernel is H $(h \cdot (gH)) = gH \quad \forall g \in G$.
2. For conjugation, where G acted on G defined by:

   $$g(h) = ghg^{-1}$$

   $g \in$ kernel of this action $\Leftrightarrow gxg^{-1} = x \quad \forall x \in G \Leftrightarrow g \in Z(G)$

Definition: An action is called **faithful** if the kernel of the action is the identity element of G.

Definition: Let X be a G-set and let $x \in X$. The **orbit** of x is defined to be:

$$Gx := \{gx \mid g \in G\}$$

Proposition: The G-orbits of X partition the set X.

Proof:
Since $x \in Gx$, $X \subseteq \bigcup_{x \in X} Gx$. So, suppose $Gx \cap Gy \neq \emptyset$, say $g_1 x = g_2 y$. Then, $x = g_1^{-1} g_2 y$, and thus $gx = (gg_1^{-1}g_2)y \in Gy$. So, $Gx \subseteq Gy$, and by symmetry $Gx = Gy$. So if they share even one point in common, then they are equal and thus a partition.

Definition: Let X be a G-set. Let $x \in X$. The **stabilizer**, denoted G_x, of x is

$$G_x := \{g \in G \mid gx = x\}$$

Remarks:

1. G_x is a subgroup of G (check this for yourself).
2. The kernel of the action is $\bigcup_{x \in X} G_x$ by definition.

Definition: An action is **transitive** if there is one orbit (which by the above theorem would be all of X).

Proposition Let X be a G-set. Let $x \in X$. Then the following equality holds.

$$|Gx| = [G : G_x]$$

and note that $|Gx|$ divides $|G|$ if G is finite.

Proof: We will do this by defining a bijective mapping from Gx to G/G_x. So define

$$f \ : \ G/G_x \to Gx$$
$$gG_x \mapsto gx$$

Claim: f is well-defined and bijective.

$$g_1 G_x = g_2 G_x \Leftrightarrow g_2^{-1} g_1 \in G_x \quad \text{(def'n of cosets)}$$
$$\Leftrightarrow g_2^{-1} g_1 x = x \quad \text{(in stabilizer)}$$
$$\Leftrightarrow g_1 x = g_2 x \quad \text{(existence of inverse)}$$

f is clearly surjective, so we have $|Gx| = [G : G_x]$.

Definition: Counting formula for G -sets: Let X be a G-set. Then

$$|X| = \sum [G : G_x]$$

Where the sum is over x in disjoint orbits.

Looking again at the translation of cosets, we had that G acted on G/H by $g(g'H) = (gg')H$. So, what do the orbits look like? We can look at the orbit of H, which is a perfectly good coset in G/H. So,

$$GH = \{gH \mid g \in G\}$$

It's easy to see that we only have one orbit, so this action is transitive. The aforementioned counting formula is not relevant here.