1. Let $E$ be the splitting field of \( f(x) = x^4 - 2 \) over $\mathbb{Q}$. We see that $E = \mathbb{Q}(\sqrt[4]{2}, i)$. Also, as $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 2$, $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and as $x^2 + 1$ is irreducible in $\mathbb{Q}(\sqrt[4]{2})$, $[\mathbb{Q}(\sqrt[4]{2}), i : \mathbb{Q}(\sqrt[4]{2})] = 2$, we have the following picture:

\[
\begin{array}{c}
\mathbb{Q}(\sqrt[4]{2}, i) \\
\downarrow 4 \\
\mathbb{Q}(\sqrt[4]{2}) \\
\downarrow 4 \\
\mathbb{Q}(i) \\
\downarrow 2 \\
\mathbb{Q}
\end{array}
\]

Therefore, we know $[E : \mathbb{Q}] = 8 = |\text{Gal}(E/\mathbb{Q})|$. Then, we may define the following maps, both fixing $\mathbb{Q}$ (hence in $G = \text{Gal}(E/\mathbb{Q})$).

\[
\sigma : E \rightarrow E \\
\sqrt[4]{2} \mapsto i\sqrt[4]{2} \\
i \mapsto i
\]

\[
\tau : E \rightarrow E \\
\sqrt[4]{2} \mapsto -\sqrt[4]{2} \\
i \mapsto -i
\]

It is also easy to see that the order of $\sigma$ is 4 and the order of $\tau$ is 2. Furthermore, $\tau\sigma(\sqrt[4]{2}) = i\sqrt[4]{2} = \sigma^3\tau(\sqrt[4]{2})$, and $\tau\sigma(i) = -i = \sigma^3\tau(i)$, so we see that $G$ behaves exactly like the dihedral group of order 8 with generators $\sigma$ corresponding to the rotations and $\tau$ corresponding to the reflections. The subgroup lattice for $G = D_8$ is below (where vertical lines represent reverse containment):

\[
\begin{array}{c}
\{1\} \\
\downarrow \\
\langle \sigma^2\tau \rangle \\
\downarrow \langle \tau \rangle \\
\downarrow \langle \sigma^2 \rangle \\
\downarrow \langle \sigma \rangle \\
\downarrow \langle \sigma^2, \tau \rangle \\
\downarrow \langle \sigma^2, \sigma\tau \rangle \\
\downarrow G
\end{array}
\]
Our next task is to find generators for each of the subfields of $E$ that correspond to the subgroups of $G$. We shall first find the generators of the fixed field of $(\tau)$. We know that $\sigma(\sqrt{2}) = \sqrt{2}$, so certainly $Q(\sqrt{2}) \subseteq E_{(\tau)}$. However, we know that $[G : (\tau)] = 4 = [E_{(\tau)} : Q]$, hence we must have that $Q(\sqrt{2}) = E_{(\tau)}$. Similarly, we have that $Q(i) = E_{(\sigma)}$. In fact, by inspection one finds most of the generators for the fields by just considering different combinations and using the obvious elements. However, finding generators for $(\langle \tau \rangle)$ and $(\langle \sigma \rangle)$ proved tricky. Using the lemma from class and applying it to $\sqrt{2}$, we get that $E_{(\sigma \tau)} = Q((i + 1)\sqrt{2})$, and $E_{(\sigma^3 \tau)} = Q((i - 1)\sqrt{2})$. The results are compiled below in the same form as the subgroup lattice that is above, so it is easy to read off the field generators corresponding to the subgroups of $G$ (where vertical lines representing containment).

2. Set $N = \{g \in G | g(K) = K\}$. Note that for any $g \in N$, $h \in H$, and $k \in K$, we have that $ghg^{-1}(k) = gh(g^{-1}(k)) = gg^{-1}(k) = k$, as $h \in \text{Gal}(E/K)$ fixes $K$. Therefore, $ghg^{-1}$ fixes $K$ and is hence in $H$, so $N \subseteq N_G(H)$. To prove the other direction, assume that there exists a $g \in N_G(H) \setminus N$, and choose a $k \in K$ such that $g(k) \notin K$. Also, choose $h \in H$ such that $h(g(k)) \neq g(k)$, which we may do assuming $|H| \geq 2$. Then we have that $g^{-1}hg(k) \neq k$, and hence $g^{-1}hg(k) \notin H$, a contradiction to $g \in N_G(H)$. Hence $N_G(H) = N$. Define a group homomorphism $\phi : N_G(H) \to \text{Aut}(K/F)$ sending $\sigma$ to $\sigma|_K$, which is well defined as $\sigma(K) = K$. Note $\phi$ is onto as any $\tau \in \text{Aut}(K/F)$ may be extended to $\sigma \in G$. However, since $\sigma|_K = \tau$, $\sigma(K) = K$, so $\sigma \in N_G(H)$. Also, $\sigma \in \ker \phi \iff \sigma|_K = \text{Id}_K \iff \sigma \in H$. Hence by the first isomorphism theorem for groups, we have that $N_G(H)/H \cong \text{Aut}(K/F)$.

3. We know that $e \in EF \cap E$ is algebraic over $K$ and hence $F$, as $K \subseteq F$. Certainly $f \in F$ is algebraic over $F$. So, as elements in $EF$ are of the form $\sum c_ia_i$, and sums, products, and quotients of algebraic elements are algebraic, we have that $EF/F$ is algebraic. Further, we can think of $EF$ as $F(E)$. As all the elements of $E$ are separable over $K$, they are certainly separable over $F$, so we are simply adjoining separable elements to a field. Thus we also see that $EF/F$ is separable. To see that $EF/F$ is normal, let $\phi : EF \to F$ be any field homomorphism fixing $F$. As $\phi$ fixes $F$, to show that $\phi(EF) = EF$, it suffices to show that $\phi(E) = E$. Thus, consider $\phi|_E : E \to \overline{F}$. $\phi$ must send algebraic elements over $K$ to other algebraic elements over $K$ in $\overline{F}$, hence $K$ and $F$ have the same algebraic closure and we may replace $\overline{F}$ with $K$ above. Hence we now have a field map $\sigma|_E : E \to \overline{K}$ fixing $F$, which in turn fixes $K$, hence as $E/K$ was normal, we have that $\sigma(E) = E$. Therefore $EF/F$ is normal, and hence Galois.
4. Let $L$ be the normal closure of $E/F$, $G = \text{Gal}(L/F)$, $H = \text{Gal}(L/E)$. Using problem 2, we have that $N_G(H) = \{g \in G | g(E) = E\}$, and $N_G(H)/H \cong \text{Aut}(E/F)$. Note that the set $S = \{\phi : E \to \bar{F}|_{\phi} = \text{Id}_F\}$ has order $p$ as $E/F$ is separable and also $\text{Aut}(E/F)$ is a subset of $S$. Also, note that $|G| \leq p!$ and also $|H| \leq (p-1)!$. Hence, it follows that if we show that $N_G(H) \neq H$ then we must have that the order of $N_G(H)/H$ is $p$, and hence $E/F$ is normal. Thus, as $E$ contains a root of $f$, it must contain all of the roots of $f$ and is hence the splitting field for $F$. Also, we know $E/F$ is Galois, and as the Galois group is of order $p$, we know that $E/F$ is cyclic. So indeed, let $\alpha_1, \ldots, \alpha_p$ be the roots of $f(x)$ and wlog, assume as above that at least $\alpha_1$ and $\alpha_2$ are in $E$. Construct a $\sigma$ as follows:

$$
\sigma : L \to L
\alpha_1 \to \alpha_2
\alpha_2 \to \alpha_1
\alpha_i \to \alpha_i \text{for } i \geq 3
$$

This $\sigma$ is in $N_G(H)$, as $\sigma(E) = E$, and also $\sigma \notin H$, as $\sigma$ does not fix $E$. Hence the above argument holds (I admit that I was a bit vague why it follows from orders that the order of $N_G(H)/H$ is $p$, but my original argument was incorrect and I had to turn it in).

5. As $G = \text{Gal}(E/\mathbb{Q})$ is a subgroup of $S_4$, and $|S_4| = 24$, we immediately have that $G = S_4$. Note that $S_3$ has index 4 in $G$, setting $H = S_3$, and $F = E_H$, we see that $[G:H] = 4 = [F: \mathbb{Q}]$. If there were a field properly between $F$ and $\mathbb{Q}$, then there would be a subgroup properly between $S_4$ and $S_3$, necessarily normal, as its index would then be 2. However, the only normal subgroup of index 2 in $S_4$ is $A_4$. Thus, $A_4$ would have to contain $S_3$, which it certainly does not. Therefore, $F = E_H$ fits the criterion of the problem.

6. It’s pretty clear that any finite normal extension of $F$ would have to contain $\sqrt{5}$, as all $F$ does not contain are the even $n$th roots of 5, all of which contain $\sqrt{5}$. Thus, in Galois language, the groups that correspond to those intermediate fields between a given finite normal extension $E$ and $F$ all must contain $\text{Gal}(E/F(\sqrt{5}))$. It then follows that $G = \text{Gal}(E/F)$ is a cyclic group, by the following lemma:

**Lemma:** Suppose there exists a proper subgroup $H$ of a group $G$ such that every proper subgroup of $G$ is contained in $H$. Then $G$ is cyclic.

**Proof:** Let $x \in G \setminus H$. Then $\langle x \rangle$ is not contained in $H$, so it cannot be proper, and thus $G = \langle x \rangle$. We now see how the cyclicity of the extension follows from the fact about $\text{Gal}(E/F(\sqrt{5}))$.

7. Let $E = \mathbb{Z}_p(t)$ where $p$ is prime and $t$ is an indeterminant over $\mathbb{Z}_p$. Let $\sigma$ be the automorphism of $E$ determined by $\sigma(t) = t + 1$ and $E_\sigma$ the fixed field of $\sigma$. Note that because $\sigma^n(t) = t + n$ for any $n \in \mathbb{N}$ and as we are over $\mathbb{Z}_p$, $\langle \sigma \rangle$ has order $p$. Also, note that $E_\sigma = E_{\langle \sigma \rangle}$. Thus, using Artin’s theorem, we see that $E/E_\sigma$ is finite and Galois. Furthermore, $\langle \sigma \rangle = \text{Gal}(E/E_\sigma)$. Also, as $\sigma(t^p - t) = t^p - t$, we certainly have that $\mathbb{Z}_p(t^p - t) \subseteq E_\sigma$. Thus, we have a picture:
Where we have that $[E : E_{\sigma}] = p$. Also, we have that $[\mathbb{Z}_p(t) : \mathbb{Z}(t^p - t)] = p$ just by looking at vector space dimension. Therefore, we must have that $[E_{\sigma} : \mathbb{Z}_p(t^p - t)] = 1$ and hence $E_{\sigma} = \mathbb{Z}_p(t^p - t)$. 