ESSENTIAL INTERSECTIONS AND VANISHING
(CO)HOMOLOGY

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Abstract. We define a class of local Noetherian rings, called essential
intersections, and show that over such rings there exist pairs of finite
modules of infinite homological dimensions which have vanishing of all
higher Ext and Tor. This generalizes a well-known phenomenon of non-
trivial vanishing of all higher Ext and Tor for pairs of finite modules
over a complete intersection of codimension at least two.

1. INTRODUCTION

Let $R$ be a local Noetherian ring which is of the form $R = Q/I$ with $Q$ a
regular local ring and $I$ an ideal in the square of the maximal ideal of $Q$.

Definition. We say that $R$ is an essential intersection if $I$ is the sum of
two non-zero ideals $I = I_1 + I_2$ of $Q$ such that $I_1 \cap I_2 = I_1I_2$.

Two ideals $I_1$ and $I_2$ having this independence property are referred to
as transversal ideals in [13], where properties of their Rees algebras were
studied. The purpose of this paper, however, is to demonstrate that essential
intersections always permit non-trivial vanishing of all higher homology and
cohomology. That is, if $R$ is an essential intersection then there exist finite
$R$-modules $M$ and $N$, both of infinite projective dimension over $R$, such
that $\text{Tor}^R_i(M, N) = 0$ for all $i \gg 0$; and there exist $R$-modules $M'$ and $N'$ of
infinite projective and injective dimensions over $R$, respectively, such that
$\text{Ext}^i_R(M', N') = 0$ for all $i \gg 0$. If we further assume that $R$ is Cohen-
Macaulay, then the modules $M'$ and $N'$ exhibiting this non-trivial vanishing
of all higher $\text{Ext}^i_R(M', N')$ may both be taken to be finite.

One may think of essential intersections as a sort of generalization of com-
plete intersections. For if $I$ is generated by a regular sequence $f_1, \ldots, f_c$,
then we have $(f_1, \ldots, f_r) \cap (f_{r+1}, \ldots, f_c) = (f_1, \ldots, f_r)(f_{r+1}, \ldots, f_c)$ for any
$1 \leq r \leq c$. The vanishing of all higher Ext and Tor over complete inter-
sections is quite well-understood today. In particular, it is well-known that
over complete intersections of codimension at least two, the vanishing of all

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higher Ext and Tor may occur non-trivially. (See, for example, Theorem 3.1 of [9], and [2].) On the other hand, in [7], [8], [6] and [12] it is shown that over hypersurfaces (= codimension 1 complete intersections), Golod rings and Gorenstein rings of low codimension, the vanishing of all higher Ext$^i_R(M,N)$ or Tor$^i_R(M,N)$ implies either $M$ or $N$ has finite projective dimension over $R$, so that there does not exist non-trivial vanishing of all higher Ext or Tor over these rings. This raises a question of the ubiquity of non-trivial vanishing of all higher homology and cohomology over local rings of a singularity more general than a complete intersection. A general study of vanishing of homology and cohomology is dependant upon first identifying a source of such rings which do admit non-trivial vanishing.

It is clear that essential intersections (which are not complete intersections) are commonplace. For example, one could take any ideal $I_1$ in the square of the maximal ideal of $Q$ such that $Q/I_1$ has positive depth. Then for an element $x$ in the square of the maximal ideal of $Q$ which is a non-zerodivisor on $Q/I_1$, the ring $Q/(x,I_1)$ is an essential intersection (and not a complete intersection if $I_1$ is not).

In Sections 2 and 3 we prove the main results on the non-trivial vanishing of all higher Ext and Tor over essential intersections. Section 4 discusses briefly some properties of essential intersections. For instance, we show that the Cohen-Macaulay and Gorenstein properties are preserved after essential intersection. We also discuss the graded case and show that low multiplicity is an obstruction to a graded affine algebra being a graded essential intersection. We also deal with the ‘abstract’ case by showing that it is enough to assume the completion of $R$ is an essential intersection in establishing non-trivial vanishing of all higher Ext and Tor. The final Section 5 gives examples and a sufficient condition for determining when a finite module over an essential intersection may participate in non-trivial vanishing of all higher Ext and Tor. This sufficient condition looks at the form of the the free resolution of the module, and is aptly implemented on the computer. We do this using the computer algebra package Macaulay 2.

Recall that the transversal property $I_1 \cap I_2 = I_1I_2$ is equivalent to the condition Tor$^i_Q(Q/I_1,Q/I_2) = 0$. Moreover, from the rigidity of Tor result for regular local rings due to Auslander [1] and Lichtenbaum [10], the vanishing of Tor$^i_Q(Q/I_1,Q/I_2)$ implies that Tor$^i_Q(Q/I_1,Q/I_2) = 0$ for all $i \geq 1$. This latter condition is what we use so often in what follows. Additionally, we tacitly rely on the fact that over a regular local ring $Q$, every finite module $M$ has projective dimension $\leq \dim Q$, which causes, for any $Q$-module $N$, Tor$^i_Q(M,N) = 0$ and Ext$^i_Q(M,N) = 0$ for all $i > \dim Q$.

Vanishing of homology is a bit less complicated, so we discuss this first.
2. Vanishing of Homology

Theorem 2.1. Let $R$ be an essential intersection. Then there exist finite $R$-modules $M$ and $N$, both of infinite projective dimension over $R$, such that $\text{Tor}_i^R(M, N) = 0$ for all $i \gg 0$.

For the proof of Theorem 2.1 it is convenient to separate out a completely elementary lemma. (See, for example, 11.51 of [11].)

Lemma 2.2. Suppose that $A$ is a commutative ring, $J$ an ideal of $A$, and set $B := A/J$.

(1) If $X$ is an $A$-module such that $\text{Tor}_i^A(X, B) = 0$ for all $i \geq 1$, then for any $B$-module $Y$ we have

$$\text{Tor}_i^A(X, Y) \simeq \text{Tor}_i^B(X \otimes_A B, Y) \quad \text{for all } i,$$

and

$$\text{Ext}_i^A(X, Y) \simeq \text{Ext}_i^B(X \otimes_A B, Y) \quad \text{for all } i.$$

(2) If $Y$ is an $A$-module such that $\text{Ext}_i^A(B, Y) = 0$ for all $i \geq 1$, then for any $B$-module $X$ we have

$$\text{Ext}_i^A(X, Y) \simeq \text{Ext}_i^B(X, \text{Hom}_A(B, Y)) \quad \text{for all } i.$$

Proof of Theorem 2.1. Since $R$ is an essential intersection, we can write $R = Q/I$ with $Q$ a regular local ring, and $I = I_1 + I_2$ an ideal in the square of the maximal ideal of $Q$ with $\text{Tor}_i^Q(Q/I_1, Q/I_2) = 0$ for all $i \geq 1$. To simplify notation, set $R_1 := Q/I_1$ and $R_2 := Q/I_2$.

Let $N'$ be any finite $R_2$-module having infinite projective dimension over $R_2$. For example, one could take the residue field $k$ of $Q$. Since $Q$ is a regular local ring we have $\text{Tor}_i^Q(R_1, N') = 0$ for all $i \gg 0$. Take an exact sequence $0 \to N'' \to R_2^\oplus \to N' \to 0$. Then from the derived long exact sequence of $\text{Tor}$

$$\cdots \to \text{Tor}_i^Q(R_1, N'') \to \text{Tor}_i^Q(R_1, R_2^\oplus) \to \text{Tor}_i^Q(R_1, N') \to \cdots,$$

and our assumption that $\text{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, we have the isomorphisms $\text{Tor}_{i+1}^Q(R_1, N') \simeq \text{Tor}_i^Q(R_1, N'')$ for all $i \geq 1$. Thus we may replace $N'$ by an $R_2$-module $N''$ of infinite projective dimension over $R_2$ such that

(1) $\text{Tor}_i^Q(R_1, N'') = 0$ for all $i \geq 1$.

Note that $R$ has finite projective dimension over $R_1$: a minimal resolution of $R$ over $R_1$ is obtained by tensoring a minimal resolution of $R_2$ over $Q$ with $R_1$. Now we let $M'$ be any finite $R_1$-module of infinite projective dimension over $R_1$ such that

(2) $\text{Tor}_i^{R_1}(M', R) = 0$ for all $i \geq 1$.

For example, one could take an appropriate syzygy over $R_1$ of $k$.

By (1) we can apply Lemma 2.2 with $A = Q$, $B = R_1$, $X = N''$ and $Y = M'$ to get $\text{Tor}_i^Q(M', N'') \simeq \text{Tor}_i^{R_1}(M', N'' \otimes_Q R_1)$. Also, by (2), we
can apply Lemma 2.2 with $A = R_1$, $B = R$, $X = M'$ and $Y = N'' \otimes_Q R_1$ to get $\text{Tor}_i^R(M', N'' \otimes_Q R_1) \simeq \text{Tor}_i^R(M' \otimes R_1, R, N'' \otimes_Q R_1)$. Putting these isomorphisms together, we have

$$\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M', N) \simeq \text{Tor}_i^Q(M', N'') = 0$$

for all $i \gg 0$, where $M := M' \otimes R_1$ and $N := N'' \otimes_Q R_1$. This establishes the claim about the vanishing of homology.

To see that both $M$ and $N$ have infinite projective dimension over $R$, note that by (2) a minimal free resolution of $M$ over $R$ is obtained by tensoring a minimal free resolution of $M'$ over $R_1$ with $R$, and this is an infinite resolution. Also, since $\text{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, by Lemma 2.2 we have $\text{Tor}_i^Q(R_1, N'') \simeq \text{Tor}_i^R(R, N'')$ for all $i$. Then by (1) we get $\text{Tor}_i^R(R, N'') = 0$ for all $i \geq 1$. Thus a minimal free resolution of $N$ over $R$ is obtained by tensoring a minimal free resolution of $N''$ over $R_2$ with $R$, and this is an infinite resolution. \hfill \qedsymbol

Remark. It is of interest to know when the vanishing of $\text{Tor}_*^R(M, N)$ begins. Having a bound on the point at which vanishing begins for all pairs of modules with eventual vanishing has powerful consequences. See, for example, [6]. In [8] the formula

$$\sup\{i \mid \text{Tor}_i^A(X, Y) \neq 0\} = \sup\{\text{depth } A_p - \text{depth } X_p - \text{depth } Y_p\},$$

where the second sup is taken over all $p \in \text{Spec } A$, was shown to hold for finite modules $X$ and $Y$ over a local ring $A$ provided one of the modules has finite projective dimension over $A$. We can use this formula to bound the point of vanishing of $\text{Tor}$ for the modules discussed in Theorem 2.1: from (1) and (2) in the proof of Theorem 2.1, and the formula above, we obtain the inequalities

$$\text{depth } Q_p - \text{depth } (R_1)_p - \text{depth } N''_p \leq 0$$

and

$$\text{depth } (R_1)_p - \text{depth } R_p - \text{depth } M'_p \leq 0,$$

which hold for any $p \in \text{Spec } Q$ and any $p \in \text{Spec } Q$ containing $I_1$, respectively. Let $q := \sup\{i \mid \text{Tor}_i^Q(M', N'') \neq 0\}$, and let $p$ be a prime ideal of $Q$ demonstrating the formula from [8] above. Then we have

$$q = \text{depth } Q_p - \text{depth } M'_p - \text{depth } N''_p \leq \text{depth } Q_p - (\text{depth } (R_1)_p - \text{depth } R_p) - (\text{depth } Q_p - \text{depth } (R_1)_p) = \text{depth } R_p.$$

Thus, for these modules from Theorem 2.1 exhibiting the non-trivial vanishing of all higher $\text{Tor}$, we know the vanishing occurs no later than $\dim R$. 

3. Vanishing of Cohomology

First we show that we have non-trivial vanishing of cohomology between finite modules in the case where \( R \) is a Cohen-Macaulay essential intersection.

**Theorem 3.1.** Suppose that \( R \) is a Cohen-Macaulay essential intersection. Then there exist finite \( R \)-modules \( M \) and \( N \), with \( M \) having infinite projective dimension over \( R \) and \( N \) having infinite injective dimension over \( R \), such that \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \).

**Proof.** Since \( R \) is the homomorphic image of a Gorenstein ring, it has a canonical module \( C \). Let \( \sim^v \) denote the dual \( \text{Hom}_R(-, C) \).

We will actually show that if \( M \) and \( N \) are finite \( R \)-modules with \( N \) maximal Cohen-Macaulay, then \( \text{Tor}^i_R(M, N) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^i_R(M, N^v) = 0 \) for all \( i \gg 0 \). Then Theorem 2.1 will provide the existence of finite \( R \)-modules \( M \) and \( N \), both of infinite projective dimension over \( R \) and \( N \) maximal Cohen-Macaulay, such that \( \text{Ext}^i_R(M, N^v) = 0 \) for all \( i \gg 0 \), and the above equivalence with \( M \) replaced by the residue field \( k \) of \( Q \) will show that \( N^v \) has infinite injective dimension over \( R \). We induct on \( d \).

For \( d = 0 \), we have by Matlis duality the isomorphisms \( \text{Tor}^R_i(M, N^v) \cong \text{Ext}^R_i(M, N^v) \) for all \( i \), which gives the desired conclusion immediately.

For \( d > 0 \), we first assume that \( M \) is also maximal Cohen-Macaulay. Choose \( x \) in the maximal ideal of \( R \) to be a non-zero divisor on \( R, M, N \) and \( N^v \). From the long exact sequence of \( \text{Tor} \) corresponding to the short exact sequence \( 0 \to N \xrightarrow{x} N \to N/xN \to 0 \) and Nakayama’s lemma we have \( \text{Tor}^i_R(M, N) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Tor}^i_R(M, N/xN) = 0 \) for all \( i \gg 0 \). Lemma 2.2 implies that \( \text{Tor}^i_R(M, N/xN) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Tor}^{i/(c)}_R(M/xM, N/xN) = 0 \) for all \( i \gg 0 \). By induction these higher \( \text{Tor} \)s vanish over \( R/(x) \) if and only if \( \text{Ext}^{i/(c)}_R(M/xM, (N/xN)^v) = 0 \) for all \( i \gg 0 \), where by \( (N/xN)^v \) we mean \( \text{Hom}_{R/(x)}(N/xN, C/xC) \).

Rees’ lemma says that \( \text{Hom}_{R/(x)}(N/xN/C/xC) \cong \text{Ext}^i_R(N/xN, C) \), and since \( N \) is maximal Cohen-Macaulay, the latter module is isomorphic to \( \text{Hom}_R(N, C)/x \text{Hom}_R(N, C) \). Thus \( (N/xN)^v \cong N^v/xN^v \). Therefore we have the equivalence \( \text{Ext}^i_R(M/xM, (N/xN)^v) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^{i/(c)}_R(M/xM, N^v/xN^v) = 0 \) for all \( i \gg 0 \). Now by Rees’ lemma again, we get \( \text{Ext}^{i/(c)}_R(M/xM, N^v/xN^v) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^i_R(M/xM, N^v) = 0 \) for all \( i \gg 0 \). By the long exact sequence of \( \text{Ext} \) associated to the short exact sequence \( 0 \to M \xrightarrow{x} M \to M/xM \to 0 \), and Nakayama’s lemma, we have \( \text{Ext}^i_R(M/xM, N^v) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^i_R(M, N^v) = 0 \) for all \( i \gg 0 \). This establishes the claim in the case where \( M \) is assumed to be maximal Cohen-Macaulay.

In general, let \( M' \) denote an appropriate syzygy over \( R \) of \( M \) which is maximal Cohen-Macaulay. Then \( \text{Tor}^i_R(M, N) = 0 \) for all \( i \gg 0 \) if and only
if \( \text{Tor}^R_1(M', N) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^1_R(M', N^\vee) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}^1_R(M, N^\vee) = 0 \) for all \( i \gg 0 \).

If we do not restrict our attention to finite modules, we can build directly modules with non-trivial vanishing cohomology over any essential intersection in a fashion analogous to Theorem 2.1.

**Theorem 3.2.** Let \( R \) be a essential intersection. Then there exists a finite \( R \)-module \( M \) of infinite projective dimension over \( R \), and an \( R \)-module \( N \) of infinite injective dimension over \( R \) such that \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \).

**Proof.** Write \( R = Q/I \) with \( I = I_1 + I_2 \) in the square of the maximal ideal of \( Q \) satisfying \( I_1 I_2 = I_1 \cap I_2 \), and set \( R_1 = Q/I_1 \) and \( R_2 = Q/I_2 \). Let \( N' \) be an \( R_2 \) module of infinite injective dimension. For example, one could take the residue field \( k \). Since \( Q \) is a regular local ring, \( \text{Ext}^i_Q(R_1, N') = 0 \) for all \( i \gg 0 \). Let \( 0 \to N' \to \mathcal{I} \to N'' \to 0 \) be an exact sequence of \( R_2 \)-modules with \( \mathcal{I} \) injective. Then we have the long exact sequence of cohomology

\[
\cdots \to \text{Ext}^i_Q(R_1, N') \to \text{Ext}^i_Q(R_1, \mathcal{I}) \to \text{Ext}^i_Q(R_1, N'') \to \cdots .
\]

Since \( \mathcal{I} \) is an injective \( R_2 \)-module, it is a direct sum of injective hulls \( E_{R_2}(R_2/p) \) of quotients \( R_2/p \) with \( p \) a prime ideal of \( R_2 \). If \( P \) is a prime ideal of \( Q \) which is a preimage of \( p \), then \( E_{R_2}(R_2/p) = \text{Hom}_Q(R_2, E_Q(Q/P)) \), where \( E_Q(Q/P) \) is the injective hull of \( Q/P \). Thus we can write \( \mathcal{I} = \text{Hom}_Q(R_2, \mathcal{J}) \) where \( \mathcal{J} \) is an injective \( Q \)-module. We have the isomorphisms \( \text{Ext}^i_Q(R_1, \text{Hom}_Q(R_2, \mathcal{J})) \simeq \text{Hom}_Q(\text{Tor}^Q_i(R_1, R_2), \mathcal{J}) \) for all \( i \) (see, for example, page 360 of [11]). Therefore we have \( \text{Ext}^i_Q(R_1, \mathcal{I}) = 0 \) for all \( i \geq 1 \), and so from the long exact sequence of Ext we get

\[
\text{Ext}^i_Q(R_1, N'') \simeq \text{Ext}^{i+1}_Q(R_1, N')
\]

for all \( i \geq 1 \). Now it is clear that we can replace \( N' \) by an \( R_2 \)-module \( N'' \) of infinite injective dimension such that

\[
(3) \quad \text{Ext}^i_Q(R_1, N'') = 0 \quad \text{for all } i \geq 1.
\]

As in the proof of Theorem 2.1, we can choose a finite \( R_1 \)-module \( M' \) of infinite projective dimension over \( R_1 \) such that

\[
(4) \quad \text{Tor}^R_i(M', R) = 0 \quad \text{for all } i \geq 1.
\]

By (3) and Lemma 2.2 we have the isomorphisms

\[
\text{Ext}^i_Q(M', N'') \simeq \text{Ext}^i_{R_1}(M', \text{Hom}_Q(R_1, N''))
\]

for all \( i \). By (4) and Lemma 2.2,

\[
\text{Ext}^i_{R_1}(M', \text{Hom}_Q(R_1, N'')) \simeq \text{Ext}^i_{R}(M' \otimes_{R_1} R, \text{Hom}_Q(R_1, N''))
\]

for all \( i \). Therefore we have

\[
\text{Ext}^i_R(M, N) \simeq \text{Ext}^i_{R_1}(M', N) \simeq \text{Ext}^i_Q(M', N'') = 0
\]

for all \( i \gg 0 \), where \( M = M' \otimes_{R_1} R \) and \( N = \text{Hom}_Q(R_1, N'') \). Hence we have vanishing cohomology.
We know from the proof of Theorem 2.1 that \( M \) has infinite projective dimension over \( R \). Therefore to finish the proof we just need to justify that \( N \) has infinite injective dimension over \( R \). Since \( \text{Tor}_i^R(R_1, R_2) = 0 \) for all \( i \geq 1 \), by Lemma 2.2 we have \( \text{Ext}_i^R(R_1, N'') \simeq \text{Ext}_i^R_2(R, N'') \) for all \( i \). Then by (3) we get \( \text{Ext}_i^R(R, N'') = 0 \) for all \( i \geq 1 \). Thus, by Lemma 2.2, a minimal injective resolution of \( N \) over \( R \) is obtained by applying \( \text{Hom}_{R_2}(R_1, -) \) to a minimal injective resolution of \( N'' \) over \( R_2 \), and this is an infinite resolution. 

\[ \square \]

Remark. Regarding the question of when the vanishing of \( \text{Ext}_i^R(M, N) \) begins, assume that \( R \) is a Cohen-Macaulay essential intersection with both \( R_1 \) and \( R_2 \) Cohen-Macaulay. In the proof of Theorem 3.1 we first assumed that \( M \) and \( N \) are maximal Cohen-Macaulay \( R \)-modules obtained from Theorem 2.1 with \( \text{Tor}_i^R(M, N) = 0 \) for all \( i \gg 0 \). The assumption of depth on \( M \) and \( N \) implies, by Auslander's formula for depth of \( \text{Tor} \) (Theorem 1.2 of [1]), that \( M' \) is a maximal Cohen-Macaulay \( R_1 \)-module and \( N'' \) is a maximal Cohen-Macaulay \( R_2 \)-module. It follows that \( \text{Tor}_i^R(M', N'') = 0 \) for all \( i \geq 1 \), and a consequence of this is that depth \( M \otimes_R N = \text{depth } R \). If \( d := \text{dim } R > 0 \) then the element \( x \) in the ‘\( d > 0 \)’ case of the proof of Theorem 3.1 may additionally be taken to be a non-zerodivisor on \( M \otimes_R N \). Chasing through the proof shows then that \( \text{Tor}_i^R(M, N) = 0 \) for all \( i \geq 1 \) implies \( \text{Ext}_i^R(M, N') = 0 \) for all \( i \geq 1 \). Therefore, without the assumption of \( M \) being maximal Cohen-Macaulay, what we know is that \( \text{Ext}_i^R(M, N') = 0 \) for all \( i > \text{dim } R \).

The point at which vanishing begins for the modules in Theorem 3.2 is more of a mystery, since \( N \) is not finitely generated.

4. Properties of Essential Intersections

We now discuss some facts about essential intersections. The first result shows how one may obtain Cohen-Macaulay and Gorenstein essential intersections. The proof of the first statement of Property (1) in the following is given in [5] (Lemma 1.10).

**Proposition 4.1.** Let \( R = Q/I \) be an essential intersection with \( I = I_1 + I_2 \) satisfying \( I_1 \cap I_2 = I_1I_2 \).

1. If \( Q/I_1 \) and \( Q/I_2 \) are Cohen-Macaulay, then so is \( R \); moreover, we have height \( I = \text{height } I_1 + \text{height } I_2 \).
2. If \( Q/I_1 \) and \( Q/I_2 \) are Gorenstein, then so is \( R \).
3. If \( Q/I_1 \) and \( Q/I_2 \) are complete intersections, then so is \( R \).

If \( Q \) contains a field, then the converses to statements (1)--(3) hold as well.

**Proof.** Set \( R_1 := Q/I_1 \) and \( R_2 := Q/I_2 \). For (1) we only need to prove the second statement. Using the Auslander-Buchsbaum formula, and the fact
that \( pd_Q R = pd_Q R_1 + pd_Q R_2 \) (see the proof for (2)) we have

\[
\text{height } I = \dim Q - \dim R \\
= \text{depth } Q - \text{depth } R \\
= pd_Q R \\
= pd_Q R_1 + pd_Q R_2 \\
= (\text{depth } Q - \text{depth } R_1) + (\text{depth } Q - \text{depth } R_2) \\
= (\dim Q - \dim R_1) + (\dim Q - \dim R_2) \\
= \text{height } I_1 + \text{height } I_2.
\]

For the proof of (2), statement (1) already establishes the necessary Cohen-Macaulay hypothesis. To finish, let \( F \) and \( G \) be minimal \( Q \)-free resolutions of \( R_1 \) and \( R_2 \), respectively. Since \( R_1 \) and \( R_2 \) are both Gorenstein, \( F \) and \( G \) are symmetric resolutions. By the vanishing of \( \text{Tor}^Q_i(R_1, R_2) = 0 \) for all \( i \geq 1 \), a \( Q \)-free resolution of \( R \) is given by \( F \otimes_Q G \), and this is again a symmetric resolution. Thus \( R \) is Gorenstein.

Statement (3) follows easily from (1) and the fact that \( \mu_Q(I) = \mu_Q(I_1) + \mu_Q(I_2) \), where \( \mu_Q(J) \) denotes the minimal number of generators of an ideal \( J \) of \( Q \).

Now suppose that \( Q \) contains a field, and that \( R \) is Cohen-Macaulay. By the intersection theorem of Peskine and Szpiro (see Corollary 9.4.6 of [3]) we have the inequality

\[
\text{depth } Q + \dim R \geq \dim R_2 + \text{depth } R_2.
\]

Therefore

\[
\text{depth } R_1 + \text{depth } R_2 = (\text{depth } Q - pd_Q R_1) + (\text{depth } Q - pd_Q R_2) \\
= \text{depth } Q + (\text{depth } Q - pd_Q R) \\
= \dim Q + \text{depth } R \\
= \dim Q + \dim R \\
\geq \dim R_1 + \text{depth } R_2.
\]

Thus \( \text{depth } R_1 \geq \dim R_1 \) and so \( R_1 \) is Cohen-Macaulay. By symmetry, so is \( R_2 \).

Now assume further that \( R \) is Gorenstein. Let \( g_1 := \text{height } I_1 \) and \( g_2 := \text{height } I_2 \). To prove the converse of (2) it suffices to show \( \text{Ext}^1_Q(R_1, Q) \simeq R_1 \) and \( \text{Ext}^2_Q(R_2, Q) \simeq R_2 \). We have by assumption that \( \text{Ext}_Q^{g_1+g_2}(R, Q) \simeq R \), and this implies that \( F_0 \simeq Q \) and \( G_0 \simeq Q \), where \( F \) and \( G \) are minimal \( Q \)-free resolutions of \( R_1 \) and \( R_2 \), respectively. It follows that \( \text{Ext}_Q^{g_1}(R_1, Q) \simeq Q/I_1 \) and \( \text{Ext}_Q^{g_2}(R_2, Q) \simeq Q/I_2 \) for ideals \( I_1 \) and \( I_2 \) of \( Q \). Now dualizing \( F \) shows that \( I_1 \subseteq I_1' \), and dualizing \( F^* := \text{Hom}_Q(F, Q) \) shows that \( I_1' \subseteq I_1 \). Similarly, \( I_2 = I_2' \).

The converse of statement (3) follows easily from the converse of (1) and the equality \( \mu_Q(I) = \mu_Q(I_1) + \mu_Q(I_2) \).
We next show that low codimension provides an obstruction to an essential intersection being Gorenstein and not a complete intersection.

**Corollary 4.2.** Suppose that $R = Q/I$ is a Gorenstein essential intersection. If height $I ≤ 3$, then $R$ is a complete intersection.

**Proof.** We have $\text{pd}_Q R = \text{pd}_Q R_1 + \text{pd}_Q R_2 ≤ 3$. Therefore $\text{pd}_Q R_1, \text{pd}_Q R_2 ≤ 2$, and since both $R_1$ and $R_2$ are Gorenstein, they must be complete intersections. Thus $R$ is a complete intersection. \qed

**Remark** Corollary 4.2 can also be proven using Theorem 3.1 and Theorem 2.3 of [12].

Example 4.4 below shows that there are Gorenstein essential intersections which are not complete intersections when height $I ≥ 4$.

The following proposition shows that the rings in examples 4.3 of [6] and 3.7 of [12] illustrating the non-trivial vanishing of Ext are in fact essential intersections.

**Proposition 4.3.** Let $(R_1, m_1, k)$ and $(R_2, m_2, k)$ be local rings which are essentially of finite type over the same field $k$. Set $m := m_1 \otimes_k m_2$. Then $(R_1 \otimes_k R_2)_m$ is an essential intersection.

**Proof.** Write $R_1 = Q_1/I_1$ and $R_2 = Q_2/I_2$ where $Q_1$ and $Q_2$ are regular local rings with maximal ideals $n_1$ and $n_2$, respectively, and $I_1$ is an ideal of $Q_1$ in $n_1^2$ and similarly for $I_2$. Set $n := n_1 \otimes_k Q_2 + Q_1 \otimes_k n_2$. Then it is easy to see that $(Q_1 \otimes_k Q_2)_n$ is a regular local ring of dimension $\dim Q_1 + \dim Q_2$.

Next, we have the natural isomorphism

$$R_1 \otimes_k R_2 \simeq (Q_1 \otimes_k Q_2)/(I_1 \otimes_k Q_2 + Q_1 \otimes_k I_2),$$

which of course holds locally as well. Therefore, in order to establish the proposition, we only need to show that

$$\text{Tor}_1^{Q_1 \otimes_k Q_2}((Q_1 \otimes_k Q_2)_n/(I_1 \otimes_k Q_2)_n, (Q_1 \otimes_k Q_2)_n/(Q_1 \otimes_k I_2)_n) = 0.$$

We will actually show that the unlocalized Tor is zero.

Let $F$ be a $Q_1$-free resolution of $Q_1/I_1$. Then by flatness over $k$, $F \otimes_k Q_2$ is exact, and so is a free resolution of $(Q_1 \otimes_k Q_2)/(I_1 \otimes_k Q_2)$ over $Q_1 \otimes_k Q_2$. Also, $F \otimes_k Q_2/I_2$ is exact. But this latter complex is isomorphic to

$$(F \otimes_k Q_2) \otimes_{(Q_1 \otimes_k Q_2)} (Q_1 \otimes_k Q_2)/(Q_1 \otimes_k I_2).$$

Thus $\text{Tor}_1^{Q_1 \otimes_k Q_2}((Q_1 \otimes_k Q_2)/(I_1 \otimes_k Q_2), (Q_1 \otimes_k Q_2)/(Q_1 \otimes_k I_2)) = 0$. \qed

**Remark.** Clearly not every essential intersection arises as in the statement of Proposition 4.3. For example, let $R$ be any one-dimensional reduced complete intersection with embedding dimension three — say for instance $(k[x, y, z]/(x^2 + y^2 + z^2, xy - z^2))_{(x, y, z)}$. Then for $(R_1 \otimes_k R_2)_m$ a one-dimensional complete intersection with embedding dimension three, either $R_1$ or $R_2$ must have nonzero nilpotents, and therefore cannot be isomorphic to $R$. 
In Lemma 3.6 of [6] it is shown that for every Gorenstein local ring \((R, m)\) with \(m^3 = 0\) which is not a complete intersection, \(\text{Tor}_i^R(M, N) = 0\) for all \(i \gg 0\) if and only if either \(M\) or \(N\) is free, for finite \(R\)-modules \(M\) and \(N\). And the same equivalence holds for Ext. Therefore there is no local Gorenstein essential intersection \((R, m)\) with \(m^3 = 0\) which is not a complete intersection. The following simple example shows that there do however exist local Gorenstein essential intersections \((R, m)\) with \(m^3 = 0\) which are not complete intersections.

**Example 4.4.** Let \(R := k[[w, x, y, z]]/(w^2, xy, xz, yz, x^2 - y^2, y^2 - z^2)\). Then \(R\) is a Gorenstein essential intersection with \((w, x, y, z)^4 = 0\).

**Graded Essential Intersections**

In this subsection we assume that \(Q := k[x_1, \ldots, x_n]\) is a polynomial ring over the field \(k\). We say that \(R\) is a graded essential intersection if \(R = Q/I\) with \(I = I_1 + I_2\) and \(I_1, I_2\) non-zero homogeneous ideals in the square of the unique homogeneous maximal ideal of \(Q\) satisfying \(I_1I_2 = I_1 \cap I_2\).

By localizing and using rigidity of Tor for regular local rings, we see that this transversal property satisfied by \(I_1\) and \(I_2\) is again equivalent to the condition \(\text{Tor}_i^Q(Q/I_1, Q/I_2) = 0\) for all \(i \geq 1\). Therefore, one may mimic the proofs of Theorems 2.1 and 3.1 verbatim in the current graded case, showing that there exist non-trivial vanishing of all higher Ext and Tor over graded essential intersections:

**Theorem 4.5.** Let \(R\) be a graded essential intersection. Then there exist finite \(R\)-modules \(M\) and \(N\), both of infinite projective dimension over \(R\), such that \(\text{Tor}_i^R(M, N) = 0\) for all \(i \gg 0\).

**Theorem 4.6.** Suppose that \(R\) is a Cohen-Macaulay graded essential intersection. Then there exist finite \(R\)-modules \(M\) and \(N\), with \(M\) having infinite projective dimension over \(R\) and \(N\) having infinite injective dimension over \(R\), such that \(\text{Ext}_i^R(M, N) = 0\) for all \(i \gg 0\).

Suggested by Theorem 3.5 of [6], we show that multiplicity plays a role in determining when a graded affine algebra is an essential intersection. We let \(e(A)\) denote the multiplicity of the graded ring \(A\).

**Proposition 4.7.** Let \(R = Q/I\) be a graded essential intersection with \(I = I_1 + I_2\) satisfying \(I_1 \cap I_2 = I_1I_2\). Set \(R_1 := Q/I_1\) and \(R_2 := Q/I_2\). Then height \(I = \text{height } I_1 + \text{height } I_2\) and \(e(R) = e(R_1)e(R_2)\).

Thus, even if both \(R_1\) and \(R_2\) have minimal possible multiplicities, \(R\) will not. See Corollary 4.9 below.

**Proof.** Let

\[ 0 \to \bigoplus_j Q(-j)^{b_{i,j}} \to \cdots \to \bigoplus_j Q(-j)^{b_{i,j}} \to R_1 \to 0 \]
be a graded free resolution of $R_1$ over $Q$. Then we have the equality

\[(6) \quad H_{R_1}(t) = \frac{S_{R_1}(t)}{(1-t)^n}, \]

where $H_{R_1}(t)$ is the Hilbert series of $R_1$, $1/(1-t)^n$ is the Hilbert series of $Q$, and $S_{R_1}(t)$ is the polynomial $\sum_{i,j}(-1)^i b_{ij} t^j$. (See, for example, 4.1.13 of [3].)

Now if we tensor the resolution in (5) with $R_2$ and use the fact that $\text{Tor}^Q_i(R_1, R_2) = 0$ for all $i \geq 1$, we obtain a graded free resolution

\[(7) \quad 0 \to \bigoplus_j R_2(-j)^{b_{0j}} \to \cdots \to \bigoplus_j R_2(-j)^{b_{0j}} \to R \to 0 \]

of $R$ over $R_2$. Therefore we have the equality

\[(8) \quad H_R(t) = S_R(t)H_{R_2}(t), \]

where $H_R(t)$ is the Hilbert series of $R$, $H_{R_2}(t)$ is the Hilbert series of $R_2$, and $S_R(t)$ is the polynomial $\sum_{i,j}(-1)^i b_{ij} t^j$. Since $S_{R_1}(t) = S_R(t)$ we may combine (6) and (8) to get

\[(9) \quad \frac{1}{(1-t)^n} H_R(t) = H_{R_1}(t) H_{R_2}(t). \]

Recall that for a graded module $A$ of dimension $d$ there exists a unique element $Q_A(t) \in \mathbb{Z}[t, t^{-1}]$ such that $H_A(t) = Q_A(t)/(1-t)^d$ and $e(A) = Q_A(1) \neq 0$. Hence we can rewrite (9) as

\[(10) \quad \frac{1}{(1-t)^n} \frac{Q_R(t)}{(1-t)^d} = \frac{Q_{R_1}(t)}{(1-t)^{d_1}} \frac{Q_{R_2}(t)}{(1-t)^{d_2}}, \]

where $d$ is the dimension of $R$, $d_1$ is the dimension of $R_1$, and $d_2$ is the dimension of $R_2$. If $n + d > d_1 + d_2$, then for $i = n + d - (d_1 + d_2) > 0$, (10) becomes $Q_R(t) = (1-t)^i Q_{R_1}(t)Q_{R_2}(t)$. But then $Q_R(1) = 0$, a contradiction. Similarly, $n + d \leq d_1 + d_2$. Hence $n + d = d_1 + d_2$, and this translates into height $I = \text{height } I_1 + \text{height } I_2$. Finally, canceling off the $(1-t)^{n+d}$ on both sides of (10), we are left with

\[Q_R(t) = Q_{R_1}(t)Q_{R_2}(t), \]

and this establishes the second claim.

In the graded case we get an analog of Proposition 4.1.

**Corollary 4.8.** Suppose that $R = Q/I$ is a graded essential intersection with $I = I_1 + I_2$ satisfying $I_1 \cap I_2 = I_1 I_2$. Then $R$ is Cohen-Macaulay if and only if both $R_1 := Q/I_1$ and $R_2 := Q/I_2$ are Cohen-Macaulay. $R$ is Gorenstein if and only if both $R_1$ and $R_2$ are Gorenstein. $R$ is a complete intersection if and only if both $R_1$ and $R_2$ are complete intersections.

**Proof.** For a graded $Q$-module $A$ we write depth $A$ to denote the length of a maximal $A$-sequence contained in the unique homogeneous maximal ideal
if \( Q \). From Proposition 4.7 we have the equality \( n + d = d_1 + d_2 \), using the notation found there. Assuming \( R \) is Cohen-Macaulay, we have
\[
\text{depth } R_1 + \text{depth } R_2 = (n - pd_Q R_1) + (n - pd_Q R_2) \\
= 2n - pd_Q R \\
= n + \text{depth } R \\
= n + d \\
= d_1 + d_2.
\]

Now since depth is always at most the dimension, we get \( \text{depth } R_1 = d_1 \) and \( \text{depth } R_2 = d_2 \).

Turning this string of equalities inside out, we get the reverse implication. For the statements regarding the Gorenstein and complete intersection hypotheses, we follow the proof of Proposition 4.1.

We can now give an obstruction, in terms of multiplicity, to Cohen-Macaulay graded affine algebras being essential intersections.

Recall that if \( Q/I \) is Cohen-Macaulay and graded, then \( e(Q/I) \geq n - \dim Q/I + 1 = \text{height } I + 1 \). And if \( Q/I \) is furthermore Gorenstein, then \( e(Q/I) \geq n - \dim Q/I + 2 = \text{height } I + 2 \).

**Corollary 4.9.** Suppose that \( R = Q/I \) is graded. If either

1. \( R \) is Cohen-Macaulay and \( e(R) < 2(\text{height } I) \), or
2. \( R \) is Gorenstein and \( e(R) < 3(\text{height } I) + 3 \),

then \( R \) is not an essential intersection.

**Proof.** Suppose on the contrary that \( R = Q/I \) is a graded essential intersection with \( I = I_1 + I_2 \) satisfying \( I_1 \cap I_2 = I_1 I_2 \). Then by the previous corollary, \( R_1 = Q/I_1 \) and \( R_2 := Q/I_2 \) are both Cohen-Macaulay. Therefore from proposition 4.7 we have
\[
e(R) = e(R_1) e(R_2) \geq (n - \dim R_1 + 1)(n - \dim R_2 + 1) \\
= (\text{height } I_1 + 1)(\text{height } I_2 + 1) \\
= \text{height } I_1 \text{ height } I_2 + (\text{height } I_1 + \text{height } I_2) + 1 \\
\geq (\text{height } I - 1) + \text{height } I + 1 \\
= 2(\text{height } I).
\]
The proof for (2) is similar.

The following easy example shows that the equality of multiplicities in Proposition 4.7 can fail in the local case.

**Example 4.10.** Let \( Q \) be the regular local ring \( k[[x, y]] \) where \( k \) is a field. Let \( I_1 := (x^2 - y^3) \) and \( I_2 := (x^2) \). Then \( I_1 \cap I_2 = I_1 I_2 \), so that \( R := Q/I_1 + I_2 \) is an essential intersection. We have \( e(Q/I_1) = e(Q/I_2) = 2 \). However, \( e(R) = 6 \).
Note that in the proof of Corollary 4.9 we only really need that \( e(R) \geq e(R_1)e(R_2) \), and Example 4.10 does not rule out this inequality.

Now we return to the local case where \( Q \) is assumed to be a regular local ring. The structure theorem for complete local rings suggests that our assumption of \( R \) being a quotient of the regular local ring \( Q \) is a natural one. We may take a more general point of view:

**Abstract Essential Intersections**

We define a local Noetherian ring \( R \) to be an abstract essential intersection if the completion \( \hat{R} \) of \( R \) is an essential intersection for some Cohen presentation \( Q/I \) of \( \hat{R} \). We have more general versions of Theorems 2.1 and Theorem 3.1.

**Theorem 4.11.** Let \( R \) be an abstract essential intersection. Then there exist finite \( R \)-modules \( M \) and \( N \), both of infinite projective dimension over \( R \), such that \( \text{Tor}_i^R(M,N) = 0 \) for all \( i \gg 0 \).

**Theorem 4.12.** Suppose that \( R \) is a Cohen-Macaulay abstract essential intersection. Then there exist finite \( R \)-modules \( M \) and \( N \), with \( M \) having infinite projective dimension over \( R \) and \( N \) having infinite injective dimension over \( R \), such that \( \text{Ext}^i_R(M,N) = 0 \) for all \( i \gg 0 \).

The proofs rely on the following observation. Assume that \( R \) is an abstract essential intersection with \( \hat{R} = Q/I \) an essential intersection. Theorems 2.1 and 3.1 provide pairs of \( \hat{R} \)-modules \( M \) and \( N \) such that \( \text{depth} M = \text{depth} \hat{R} \) (take syzygies if necessary) and all higher \( \text{Tor}_i^R(M,N) \) and \( \text{Ext}_i^R(M,N^\vee) \) vanish non-trivially. To simplify notation in the following, replace \( N^\vee \) by \( N \). Let \( \mathfrak{x} = x_1, \ldots, x_d \) be a maximal regular sequence on both \( M \) and \( N \). Then looking successively at the long exact sequences of \( \text{Ext} \) coming from the short exact sequences

\[
0 \to N/(x_1, \ldots, x_{i-1})N \xrightarrow{\iota_i} N/(x_1, \ldots, x_i)N \to N/(x_1, \ldots, x_i)N \to 0,
\]

we have \( \text{Ext}_R^i(M,N) = 0 \) for all \( i \gg 0 \) implies \( \text{Ext}_R^i(M,N/(\mathfrak{x})N) = 0 \) for all \( i \gg 0 \). Similarly, the short exact sequences

\[
0 \to M/(x_1, \ldots, x_{i-1})M \xrightarrow{\iota_i} M/(x_1, \ldots, x_i)M \to M/(x_1, \ldots, x_i)M \to 0
\]

give \( \text{Ext}_R^i(M,N/(\mathfrak{x})N) = 0 \) for all \( i \gg 0 \) implies \( \text{Ext}_R^i(M/(\mathfrak{x})M,N/(\mathfrak{x})N) = 0 \) for all \( i \gg 0 \). Now \( M/(\mathfrak{x})M \) and \( N/(\mathfrak{x})N \) are \( \hat{R} \)-modules of finite length, and therefore they are finite \( R \)-modules. By faithful flatness we have the equivalence \( \text{Ext}_R^i(M/(\mathfrak{x})M,N/(\mathfrak{x})N) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Ext}_R^i(M/(\mathfrak{x})M,N/(\mathfrak{x})N) = 0 \) for all \( i \gg 0 \).

It remains to see that \( M/(\mathfrak{x})M \) and \( N/(\mathfrak{x})N \) have infinite projective and injective dimensions over \( R \), respectively. But this follows easily from faithful flatness and the short exact sequences above.

The proof for the vanishing of all higher Tors is similar.
5. Examples and a Sufficient Condition

Set $R_1 = Q/I_1$ and $R_2 = Q/I_2$ with $Q$ a regular local ring (or a polynomial ring) and $I = I_1 + I_2$ non-zero (homogeneous) ideals in the square of the (homogeneous) maximal ideal of $Q$ satisfying $I_1 \cap I_2 = I_1I_2$. In this section we discuss a sufficient condition for determining whether an $R$-module $M$ has a syzygy over $R$ of the form $M' \otimes_{R_1} R$ for some $R_1$-module $M'$ of infinite projective dimension over $R_1$ satisfying $\text{Tor}_i^{R_1}(M', R) = 0$ for all $i \geq 1$ (and hence of the form identified in (2) in the proof of Theorem 2.1).

Note that if we find a complementary $R$-module $N$ with a syzygy over $R$ of the form $N'' \otimes_{R_2} R$ with $N''$, an $R_2$-module of infinite projective dimension over $R_2$ satisfying $\text{Tor}_i^{R_2}(R, N'') = 0$ for all $i \geq 1$, then by Lemma 2.2, $\text{Tor}_i^Q(R_1, N'') = 0$ for all $i \geq 1$. This is (1) in the proof of Theorem 2.1, and so for these modules $M$ and $N$ we achieve non-trivial vanishing of all higher $\text{Tor}_i^R(M, N)$.

Let

$$\mathbf{F} : \cdots \to F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \to M \to 0$$

be a minimal $R$-free resolution of $M$. Choose a sequence of free $Q$-modules $\widetilde{F}_i$ and maps $\widetilde{\delta}_i$ between them

$$\widetilde{\mathbf{F}} : \cdots \to \widetilde{F}_2 \xrightarrow{\widetilde{\delta}_2} \widetilde{F}_1 \xrightarrow{\widetilde{\delta}_1} \widetilde{F}_0 \to 0$$

such that $\mathbf{F}$ and $\widetilde{\mathbf{F}} \otimes_Q R$ are isomorphic complexes. It is useful to think of the maps $\delta_i$ as being given by matrices over $R$ (with respect to some fixed bases of the $F_i$), in which case the maps $\widetilde{\delta}_i$ may be thought of as matrices of preimages in $Q$ of the entries of the matrices representing the $\delta_i$. Generally $\widetilde{\delta}_{i-1} \widetilde{\delta}_i \neq 0$. However, since $\mathbf{F}$ is a complex of $R$-modules, $\widetilde{\delta}_{i-1} \widetilde{\delta}_i \equiv 0$ modulo $I$.

For the sufficient condition given below we will be considering the sequences of maps

$$(11) \quad \widetilde{F}_i \otimes_Q R_j \xrightarrow{\widetilde{\delta}_i \otimes R_j} \widetilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\widetilde{\delta}_{i-1} \otimes R_j} \widetilde{F}_{i-2} \otimes_Q R_j$$

For $j = 1, 2$ and $i \geq 2$.

**Proposition 5.1.** Let $M$ be a finite $R$-module of infinite projective dimension over $R$, and suppose $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{\delta}})$ is some lifting to $Q$ of an $R$-free resolution $(\mathbf{F}, \delta)$ of $M$. If the sequence of maps (11) forms an exact sequence for some $i \geq 2$, then $M$ has a syzygy over $R$ of the form $M' \otimes_Q R_j$ where $M'$ is an $R_j$ module satisfying $\text{Tor}_i^{R_j}(M', R) = 0$ for all $l \geq 1$, and hence $M$ participates in a non-trivial vanishing of all higher Tor.

**Proof.** Without loss of generality assume that $j = 1$, and that (11) forms an exact sequence for fixed $i \geq 2$. Let $M_{i-2} := \text{coker}(\widetilde{\delta}_{i-1} \otimes R_1)$. Then

$$\widetilde{F}_i \otimes_Q R_1 \xrightarrow{\widetilde{\delta}_i \otimes R_1} \widetilde{F}_{i-1} \otimes_Q R_1 \xrightarrow{\widetilde{\delta}_{i-1} \otimes R_1} \widetilde{F}_{i-2} \otimes_Q R_1 \to M_{i-2} \to 0$$
is the beginning of an $R_1$-free resolution of $M_{i-2}$. If we tensor this complex with $R$ we get back down to the complex $F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} F_{i-2}$, which is exact. This means that $\text{Tor}_1^R(M_{i-2}, R) = 0$. By Lemma 2.2 we have $\text{Tor}_l^R(M_{i-2}, R) \simeq \text{Tor}_l^Q(M_{i-2}, R_2)$ for all $l \geq 1$. Therefore, by rigidity of Tor for regular local rings, $\text{Tor}_l^R(M_{i-2}, R) = 0$ for all $l \geq 1$. This finishes the proof, since $M_{i-2} \otimes_{R_1} R \simeq M_{i-2} := \text{coker} \partial_{i-1}$ is a syzygy of $M$ over $R$. \hfill\Box

**Remark.** Suppose $j = 1$. If $I_2$ happens to be generated by a $Q$-regular sequence, then we *a priori* only need to know that the sequence of maps in (11) with $j = 1$ forms a *complex* in order to invoke the conclusion of Proposition 5.1. For if (11) forms a complex with $j = 1$, and $I_2$ is generated by a $Q$-regular sequence, then this sequence is also regular on $R_1$, and by working our way inductively from $R$ up to $R_1$, Nakayama’s lemma yields that (11) is exact.

Next we give examples using Macaulay 2 which illustrate Proposition 5.1. We first discuss a few details of the liftings ($\tilde{F}, \tilde{\partial}$), and define special maps related to the notion of Eisenbud operators, which were developed in [4] for finite modules over a complete intersection.

Fix a minimal generating set $f_1, \ldots, f_c$ of $I$, and assume that $I_1$ is generated by $f_1, \ldots, f_r$, and $I_2$ generated by $f_{r+1}, \ldots, f_c$ for $1 \leq r \leq c - 1$. Since the products $\tilde{\partial}_{i-1} \tilde{\partial}_i$ are zero modulo $I = I_1 + I_2$, we may express them in terms of the $f_j$: write

$$
\tilde{\partial}_{i-1} \tilde{\partial}_i = \sum_{j=1}^c f_j \tilde{t}_{i,j},
$$

where the $\tilde{t}_{i,j}$ are maps $\tilde{t}_{i,j} : \tilde{F}_i \rightarrow \tilde{F}_{i-2}$. Note that these maps are not uniquely defined. They depend (first on the resolution $F$, then on the lifting $(\tilde{F}, \tilde{\partial})$, and then) on the choice of the expression in (12).

In investigating when the sequence (11)

$$
\tilde{F}_i \otimes_Q R_j \xrightarrow{\partial_i \otimes R_j} \tilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\partial_{i-1} \otimes R_j} \tilde{F}_{i-2} \otimes_Q R_j
$$

is exact, we proceed in two steps. First we need to know when it forms a complex. This is implied by the condition

$$
\tilde{t}_{i,r+1} \otimes R_1 = \cdots = \tilde{t}_{i,c} \otimes R_1 = 0
$$

when $j = 1$ and

$$
\tilde{t}_{i,1} \otimes R_2 = \cdots = \tilde{t}_{i,r} \otimes R_2 = 0
$$

when $j = 2$, for some choice of the $\tilde{t}_{i,j}$ defined by the expression in (12). Once we know conditions (13) or (14) hold, we compute the homology of the corresponding complex (11) to see that it is zero.

In the following examples, we perform both steps using Macaulay 2. For the first step we use a special script we provide called getEisoplist which
computes the maps $\tilde{t}_{i,j}$ and stores them as a list of lists called Eisoplist. Because the internal indexing used by Macaulay 2 starts at 0, the element Eisoplist\#i\#j actually represents the map $\tilde{t}_{i+2,j+1}$. The code for this script is given in the appendix. We remark here that the code is also able to compute the Eisenbud operators, which are simply the maps $t_{i,j} \coloneqq \tilde{t}_{i,j} \otimes_{Q} R$ (defined in [4] in the case where $R$ is a complete intersection).

The input for this script is a chain complex and an integer. Presumably, the chain complex is a free resolution $(F, \partial)$ over $R$ of the module $M$, and this may be obtained simply by using the res command in Macaulay 2. The integer tells the script up to what degree $i$ the maps $\tilde{t}_{i,j}$ should be computed. The lifting $(\tilde{F}, \tilde{\partial})$ of the given resolution $(F, \partial)$ of is done in the script using the Macaulay 2 command lift. Finally, the choice of the $\tilde{t}_{i,j}$ defined by expression (12) is decided in the script using the //Igb command in Macaulay 2, where Igb is a Gröbner basis of the ideal $(f_1, \ldots, f_c)$ with the ChangeMatrix => true option: we need a Gröbner basis of $(f_1, \ldots, f_c)$ in order to express a given element of $(f_1, \ldots, f_c)$ in terms of this Gröbner basis, and the ChangeMatrix => true option tells Macaulay 2 to remember how each Gröbner basis element is expressed in terms of the $f_i$, so that altogether elements of $(f_1, \ldots, f_c)$ are expressed in terms of the $f_i$. This decomposition is performed by Macaulay 2 a row at a time.

**Example 5.2.** Let $Q \coloneqq \mathbb{Q}[x,y,z]$ and $R \coloneqq Q/I$, where

$$I \coloneqq (x^2 - yz, xz - y^2, z^2 - xy, x^2 + yz).$$

Then $R$ is a zero-dimensional essential intersection, and the $R$-module $M \coloneqq R/(x + y + z)$ participates in non-trivial vanishing of all higher $\text{Tor}$.

We first load the script getEisoplist, show that $R$ is in fact an essential intersection by testing $\text{Tor}^Q_1(Q/I_1, Q/I_2) = 0$, then exhibit a minimal resolution of $M$, showing that it has infinite projective dimension over $R$.

```plaintext
i1 : load"getEisoplist.m2"
-- loaded getEisoplist.m2

i2 : Q = QQ[x,y,z];
i3 : I = ideal(x^2-y*z,x*z-y^2,z^2-x*y,x^2+y*z);

i4 : Tor_1(coker matrix{{x^2-y*z,x*z-y^2,z^2-x*y}},coker matrix{{x^2+y*z}}) == 0
  o4 = true

i5 : R = Q/I

i6 : M = coker matrix{{x+y+z}}
```
\( o6 = \text{cokernel} \mid x+y+z \mid \)

\[
\begin{array}{cccccc}
1 & 1 & 2 & 4 & 8 & 16 & 32 \\
\end{array}
\]

\( o6 : \text{R-module, quotient of R} \)

\( i7 : \text{Mres} = \text{res}(R,\text{LengthLimit}=6) \)

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 8 & 16 & 32 \\
\end{array}
\]

\( o7 = R \leftarrow R \leftarrow R \leftarrow R \leftarrow R \leftarrow R \leftarrow R \\
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
\]

\( o7 : \text{ChainComplex} \)

Now we compute the maps \( \tilde{t}_{i,j} \). What is shown is \( \{\tilde{t}_{2,1}, \tilde{t}_{2,2}, \tilde{t}_{2,3}, \tilde{t}_{2,4}\} \). Notice that \( \tilde{t}_{2,4} = 0 \), and so it is also zero modulo \( I_1 \).

\( i8 : \text{MEisolist} = \text{getEisolist}(\text{Mres},2) \)

\( o8 = \{\{2\} | 0 \ 1 \ 1, \{2\} | -1 \ 0 \ 0, \{2\} | -1 \ -1 \ 0\} \)

\( o8 : \text{List} \)

The next step is check that the homology of the complex

\[
\tilde{F}_2 \otimes_Q R_1 \xrightarrow{\tilde{g}_2 \otimes R_1} \tilde{F}_1 \otimes_Q R_1 \xrightarrow{\tilde{g}_1 \otimes R_1} \tilde{F}_0 \otimes_Q R_1
\]

is zero (although we do not really need this step, by the remark following Proposition 5.1, since \( I_2 = (x^2 + yz) \) is generated by a regular element). First we need to define the ring \( R_1 \).

\( i9 : \text{use Q} \)

\( o9 = Q \)

\( o9 : \text{PolynomialRing} \)

\( i10 : R1 = Q/\text{ideal}(x^2 - y^2, x^2 y, y^2, x^2 - x + y) \)

\( o10 = R1 \)

\( o10 : \text{QuotientRing} \)

\( i11 : \text{homology}(\text{lift(Mres.dd,1,Q)} ** R1,\text{lift(Mres.dd,2,Q)} ** R1) == 0 \)

\( o11 = \text{true} \)

Therefore, by Proposition 5.1, \( M \) participates in non-trivial vanishing.

We can build a companion module \( N \) for \( M \) as per Theorem 2.1, which yield non-trivial vanishing of all higher \( \text{Tor}_i^R(M,N) \). The steps are: define \( R_2 \), resolve the residue field over this ring, take an appropriate syzygy, and tensor this syzygy down to the ring \( R \).

\( i12 : \text{use Q}; \)
i13 : R2 = Q/ideal(x^2+y^z)

o13 = R2

o13 : QuotientRing

i14 : Nres = res coker vars R2

\[ \begin{array}{ccccccc}
1 & & & & & & \\
& 3 & & & & & 4 \\
& & 4 & & & & 4 \\
& & & 4 & & & 4 \\
\end{array} \]

o14 = R2 \leftarrow R2 \leftarrow R2 \leftarrow R2 \leftarrow R2

0 1 2 3 4

o14 : ChainComplex

i15 : N = (coker lift(Nres.dd_3, Q)) ** R

o15 = cokernel {2} | x y 0 x |
   {2} | -y x 0 -y |
   {2} | z 0 x 0 |
   {2} | 0 z y x |

\[
\begin{array}{cccc}
& & & 4 \\
4 & & & \\
& & & \\
& & & \\
\end{array}
\]

o15 : R-module, quotient of R

The beginning of a minimal resolution of N over R is given to show that
pd_R N = \infty. Afterwards we compute \{t_{2,1}, t_{2,2}, t_{2,3}, t_{2,4}\} for N. Note that
\tilde{t}_{2,1} = \tilde{t}_{2,2} = \tilde{t}_{2,3} = 0

i16 : Nres = res(N, LengthLimit=>6)

\[ \begin{array}{ccccccc}
4 & & & & & & \\
& 4 & & & & & 4 \\
& & 4 & & & & 4 \\
& & & 4 & & & 4 \\
& & & & 4 & & 4 \\
\end{array} \]

o16 = R \leftarrow R \leftarrow R \leftarrow R \leftarrow R \leftarrow R \leftarrow R

0 1 2 3 4 5 6

o16 : ChainComplex

i17 : NEisoplist = getEisoplist(Nres, 2)

o17 = {{0, 0, 0, \{2\} | 1 0 0 1 |}
   {2} | 0 1 0 0 |}
   {2} | 0 0 1 0 |}
   {2} | 0 0 0 1 |}

o17 : List

Finally, we compute the homology of the corresponding complex to show
that it is zero. We also show that indeed the first several Tor^R(M, N) are
zero.

i18 : homology(lift(Nres.dd_1, Q) ** R2, lift(Nres.dd_2, Q) ** R2) = 0

o18 = true
Example 5.3. Let $Q := \mathbb{Q}[u, v, w, x, y, z]$ and $R := Q/I$, where

$$I := (uv - vx, uw - uz - wx + xz, vw - vz, u^2 - v^2 - 2ux + x^2, \quad v^2 - w^2 + 2wz - z^2, xy - vx, xz, yz - vz, \quad x^2 - y^2 + 2vy - v^2, v^2 + y^2 - 2vy - z^2).$$

Then $R$ is a zero-dimensional Gorenstein essential intersection.

If we split up $I$ with $I_1$ being generated by the first five generators of $I$ and $I_2$ generated by the second five, then we exhibit that $R$ is an essential intersection.

The following resolution of $I$ over $Q$ shows that in fact $R$ is Gorenstein.

Next we identify a module $M$ which participates in non-trivial vanishing. We compute the $\tilde{t}_{i,j}$ for $M$ and show that $\tilde{t}_{2,6} = \tilde{t}_{2,7} = \tilde{t}_{2,8} = \tilde{t}_{2,9} = \tilde{t}_{2,10} = 0$. Then we show that the corresponding complex $\tilde{F}_2 \otimes_Q R_1 \xrightarrow{\tilde{\partial}_2 \otimes R_1} \tilde{F}_1 \otimes_Q R_1 \xrightarrow{\tilde{\partial}_1 \otimes R_1} \tilde{F}_0 \otimes_Q R_1$ has zero homology.
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i26 : Mres = res(M,LengthLimit=>6)

\[
\begin{array}{cccccc}
1 & 3 & 8 & 21 & 55 & 144 & 377 \\
\end{array}
\]

o26 = R <-- R <-- R <-- R <-- R <-- R <-- R

0 1 2 3 4 5 6

o26 : ChainComplex

i27 : tM = getEisolist(Mres,2);

i28 : tM#0#5,tM#0#6,tM#0#7,tM#0#8,tM#0#9

o28 = (0, 0, 0, 0, 0)

o28 : Sequence

i29 : use Q;

i30 : R1 = Q/ideal(u*v-v*x,u*w-w*z-v*x+v*z,v*v-v*z,
  \quad u^2-v^2-2u*v-x^2,v^2-v^2+2v*z-z^2);

i31 : homology(lift(Mres.dd_1,Q) == R1,lift(Mres.dd_2,Q) == R1) == 0

o31 = true

Now we identify a companion module $N$ for $M$ such that the pair has non-
trivial vanishing of all higher Tor. We compute the $\tilde{t}_{i,j}$ for $N$ and show that
\[
\tilde{t}_{2,1} = \tilde{t}_{2,2} = \tilde{t}_{2,3} = \tilde{t}_{2,4} = \tilde{t}_{2,5} = 0.
\]
Then we show that the corresponding complex has zero homology.

i32 : use R;

i33 : N = coker matrix{{x,y-v,z}};

i34 : Nres = res(N,LengthLimit=>6)

\[
\begin{array}{cccccc}
1 & 3 & 8 & 21 & 55 & 144 & 377 \\
\end{array}
\]

o34 = R <-- R <-- R <-- R <-- R <-- R <-- R

0 1 2 3 4 5 6

o34 : ChainComplex

i35 : tN = getEisolist(Nres,2);

i36 : tN#0#0,tN#0#1,tN#0#2,tN#0#3,tN#0#4

o36 = (0, 0, 0, 0, 0)

o36 : Sequence

i37 : use Q;

i38 : R2 = Q/ideal(x*y-v*x,x*z-v*z,x^2-y^2+2v*y-v^2,v^2+2v*x-y^2,z^2);
Finally, we compute the first few Tors, and then following Theorem 3.1, we show that the first few $\text{Ext}^R_m(M, \text{Hom}_R(N, R))$ vanish.

i40 : Tor_1(M, N) == 0, Tor_2(M, N) == 0, Tor_3(M, N) == 0

o40 = (true, true, true)
o40 : Sequence

i41 : Hom(N, R)
o41 = image | z2 |

1

o41 : $R$-module, submodule of $R$
i42 : $\text{Ext}^1_m(M, \text{Hom}(N, R)) == 0, \text{Ext}^2_m(M, \text{Hom}(N, R)) == 0, \text{Ext}^3_m(M, \text{Hom}(N, R)) == 0

o42 = (true, true, true)
o42 : Sequence

APPENDIX

Here is the Macaulay 2 script getEisolist.m2. The script can also be used to compute the Eisenbud operators in the case where $R$ is a complete intersection by simply adding the line(s)

--- tensor $t_{(ij)}$ with $R$ (= ring $N$)
tempCoef = substitute(tempCoef, ring $N$);

immediately following the line

1 := 0;

near the end.

g Eisolist = method()
g Eisolist (ChainComplex, ZZ) := (Nres, n) -> (

--- Input: A free resolution of a module over a graded quotient
--- ring and an integer indicating how far the
--- Eisenbud operators should be constructed.
--- Output: A List of Lists of Matrices, which contain the generalized
--- Eisenbud operators $t_{(ij)}$.
--- get the module that the ChainComplex is a resolution of
N := cokernel Nres.dd_1;
--- get the ring that the module is over
R := ring N;
--- get the ring that $R$ is a quotient of
Q := ambient R;
--- n is the highest degree for which the generalized Eisenbud
--- operators are constructed
firstMatList := 0;
i := 1;
--- the below loop builds a list of matrices which have entries
--- in R and are the liftings of differentials of the resolution
while (i <= n) do
{
  firstMatrList = firstMatrList | {lift(Nres.dd_i, 0)};
  i = i + 1;
};
secondMatrList := {};
i = 0;
--- the below loop takes the entries of firstMatrList and composes
--- the maps pairwise. Using the n above, this creates a list of
--- length n-1.
while (i < n-1) do
{
  secondMatrList =
  secondMatrList | {firstMatrList#i * firstMatrList#(i+1)};
  i = i + 1;
};
--- create a groebner basis for the ideal that we are modding out
--- in the ring of our module N
I := ideal N;
--- must use ChangeMatrix => true for division to be used
Gb := gb(I, ChangeMatrix => true);
i = 0;
Eisoplist := {};
--- the below loop builds a list of lists of matrices. Each
--- entry in Eisoplist is a list of matrices whose length is
--- the number of generators of the ideal we are modding out by.
--- Since each product matrix is congruent to zero mod the f's,
--- we can represent the entries in terms of the generators
--- of the ideal. We decompose each matrix to the form
--- (matrix_1)f1 * ... * (matrix_c)f_c, where c is the number of
--- generators of the ideal we are modding out by. Thus, when you
--- see Eisoplist#i#j below, it represents t_(ij), the f_jth component
--- of the ith matrix in the above secondMatrList
while (i < n-1) do
{
  j := 0;
  --- had to be a Mutable List to access and change elements
  tempMatrList := new MutableList from {};
  while (j < numgens target secondMatrList#i) do
  {
    tempRow := secondMatrList#i"{j};
    --- the below command inputs a row of a matrix and
    --- outputs a matrix whose columns are the components
    --- of the elements of the input matrix in the original
    --- basis of I.
    tempCoef := tempRow // Gb;
    --- if we are on the first row, we need to insert the
    --- matrix in the list, otherwise, just append to the
    --- one that we're on.
    l := 0;
    while (l < numgens I) do
    {
      if (j == 0) then
      {
        tempMatrList = append(tempMatrList, tempCoef"{l});
      }
      else
      {
        tempMatrList = append(tempMatrList, tempCoef+"{l});
      }
    }
\begin{verbatim}
    (    tempMatrList@1 = tempMatrList@1 || tempCoef TCP
    );
    l = l + 1;
    );
    j = j + 1;
    );
    i = i + 1;
    Eisoplist = Eisoplist | {toList(tempMatrList)};
    );
    Eisoplist
\end{verbatim}

REFERENCES


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