Relating minimum degree and the existence of a $k$-factor

Stephen G. Hartke∗, Ryan Martin† and Tyler Seacrest‡

October 6, 2010

Abstract

A $k$-factor in a graph $G$ is a spanning regular subgraph in which every vertex has degree exactly $k$. If a graph $G$ on $n$ vertices has minimum degree $\delta \geq n/2$ and $k$ is a positive integer such that $kn$ is even and $k < (\delta + \sqrt{2\delta n - n^2} + 8)/2$, then $G$ has a $k$-factor. Moreover, this is within an additive constant of 1 of being best possible.

1 Introduction

The classic result of Dirac [5] states that any simple graph on $n \geq 3$ vertices with minimum degree at least $n/2$ has a Hamiltonian cycle. This result has been generalized in a number of ways, including where the minimum degree being larger than $n/2$ gives extra structure beyond a Hamiltonian cycle. For example, Pósa (in the case $k = 2$, see Erdős [6]) and Seymour [13] conjectured that a graph of minimum degree $k/k+1$ contains the $k$th power of a Hamiltonian cycle, and Komlós, Sárközy, and Szemerédi [9] proved an approximate version of this conjecture.

Katerinis [8] proved that a graph with minimum degree at least $n/2$ has a large $k$-factor, where a $k$-factor, for any nonnegative integer $k$ is simply a $k$-regular spanning subgraph.

Theorem 1 (Katerinis [8]). Let $G$ be a simple graph on $n \geq 2$ vertices with minimum degree at least $n/2$. Then $G$ contains a $k$-factor for any $k \leq n + 5/4$ such that $kn$ is even.

In this note, we show that if the minimum degree is larger than $n/2$, then a denser $k$-factor exists. Our main theorem is as follows:

∗Department of Mathematics, University of Nebraska, Lincoln, NE 68588. hartke@math.unl.edu. This author’s research partially supported by a Nebraska EPSCoR First Award and National Science Foundation grant DMS-0914815.

†Department of Mathematics, Iowa State University, Ames, IA 50011. rymartin@iastate.edu. This author’s research partially supported by NSF grant DMS-0901008 and by an Iowa State University Faculty Professional Development grant.

‡Department of Mathematics, University of Nebraska, Lincoln, NE 68588. s-tseacre1@math.unl.edu.
Theorem 2. Let $G$ be a graph with $n$ vertices, let $\delta = \delta(G)$, and suppose $\delta \geq n/2$. Set

$$\rho = \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2}.$$ 

Then $G$ has a $k$-factor for any $k < \rho$ such that $kn$ is even. In particular, $G$ has a $k$-factor for some $k \geq \rho - 2$.

Note that for $\delta = n/2$, one obtains a $k$ factor for any $k$ up to $\frac{n+2\sqrt{n}}{4} > \frac{n+5}{4}$, so Theorem 2 generalizes Katerinis’ result. As $\delta$ gets closer to $n$, we see that $k$ approaches $n$ as well. See Figure 1.

Furthermore, we show that Theorem 2 is almost best possible.

Theorem 3. Let $n$ and $\delta$ be positive integers such that $n/2 \leq \delta \leq n - 1$. There exists a graph $G$ on $n$ vertices with minimum degree $\delta$ such that if

$$k > \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4}$$

then $G$ has no $k$-factor.

The case of $\delta = n/2$ was established by Katerinis [8]. Summarizing Theorems 2 and 3 for when $\delta > n/2$,

Corollary 4. Given positive integers $n$ and $\delta$ such that $n/2 < \delta \leq n - 1$, then every graph $G$ on $n$ vertices with minimum degree at least $\delta$ has a $k$-factor for any $k$ such that $kn$ is even and $k < \rho$. Furthermore, there exists a graph for which there is no $k$-factor for $k > \rho + 2/\sqrt{n} + 8$.

Thus, the only values of $k$, with $kn$ even, where we do not know if every graph on $n$ vertices with minimum degree $\delta$ contains a $k$ factor is when the value $k$ lands in the range $[\rho, \rho + 2/\sqrt{n} + 8]$. Hence our bounds are off by at most an additive constant of 1.

The rest of the note is laid out as follows: Section 2 gives some motivation and origins to this problem. Section 3 states the main tools we will use to prove the theorems. Section 4 is the proof of Theorem 2 modeled on the ideas of Katerinis’ result. Section 5 is the proof of Theorem 3 and section 6 is the proof of Corollary 4.

The authors, after finishing their work, were subsequently informed that our main result was proven in a manuscript by D. Christofides, D. Kühn and D. Osthus [1] that is, as yet, unpublished.

2 Motivation

This problem was originally motivated by the so-called multipartite version of the Hajnal-Szemerédi theorem. A version of Hajnal and Szemerédi’s famous theorem [7] is as follows:

Theorem 5 (Hajnal-Szemerédi). Let $k \geq 2$ be a positive integer. If $G$ is a simple graph on $n$ vertices minimum degree at least $(k - 1)n/k$, then $G$ contains a subgraph consisting of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$. 2
Figure 1: This plot shows how large a $k$-factor exists (if $kn$ is even) for a given minimum degree.

(The case for $k = 2$ is a consequence of Dirac’s theorem and $k = 3$ was originally proved by Corrádi and Hajnal [2].) Theorem 5 is best possible in general. For the example where $n$ is divisible by $k$, we may consider a complete $k$-partite graph with one class of size $n/k - 1$, another of size $n/k + 1$ and the rest of size $n/k$. Such a graph has minimum degree $(k - 1)n/k - 1$ but no subgraph consisting of $n/k$ vertex disjoint copies of $K_k$.

The multipartite version is as follows:

Conjecture 6. Let $k \geq 2$ be a positive integer. If $G$ is a $k$-partite graph with $N$ vertices in each part and each vertex is adjacent to at least $(k - 1)N/k$ vertices in each of the other parts, then either $G$ contains a subgraph consisting of $N$ vertex-disjoint copies of $K_k$ or $kN$ is odd, $N$ is a multiple of $k$ and $G$ is isomorphic to a single graph with the property that each vertex is adjacent to exactly $(k - 1)N/k$ vertices in each of the other parts.

For $k = 2$, it is merely a consequence of König-Hall. For $N$ sufficiently large, the conjecture has been proven for $k = 3, 4$ [10, 11] and various Erdős-Stone-type generalizations have been proven [16, 12]. Csaba and Mydlarz [4] were able to prove an approximate version. To wit:

Theorem 7 (Csaba-Mydlarz). Let $q \geq 5$ be a positive integer. If $G$ is a $q$-partite graph with $N$ vertices in each part and each vertex is adjacent to at least $(k - 1)N/k$ vertices in each of the other parts, where $k = q + O(\log q)$, then $G$ contains a subgraph consisting of $N$ vertex-disjoint copies of $K_k$, as long as $N$ is sufficiently large.

The main tool they use is a lemma of Csaba [3]:

\[
\delta/n + \sqrt{2\delta/n - 1} = \frac{1}{2}, \quad \frac{1}{4}
\]

\[
\frac{\rho}{n} = \frac{1}{2}, \quad \frac{1}{4}
\]
**Theorem 8** (Csaba). Let $G$ be a bipartite graph where each part has size $n$, with minimum degree $\delta = \delta(G)$, and assume $\delta \geq n/2$. Set

$$\rho = \delta + \sqrt{2\delta n - n^2}.$$ 

Then $G$ contains a $\lfloor \rho \rfloor$-factor.

Theorem 2 is, therefore, the general graph version of Theorem 8.

### 3 Main tools

A major tool in finding $k$-factors is Tutte’s $f$-factor theorem. Given a graph $G$ with a non-negative function $f : V(G) \to \mathbb{Z}$, $G$ has an $f$-factor if $G$ has a spanning subgraph in which vertex $v$ has degree $f(v)$.

Tutte [14, 15] gave necessary and sufficient conditions that characterized when a graph has a $f$-factor. Here, for a set of vertices $S$ we write $f(S)$ to denote $\sum_{v \in S} f(v)$. For two sets of vertices $S, T \subseteq V(G)$, $e(S, T)$ denotes the number of edges with one endpoint in $S$ and the other endpoint in $T$. For disjoint $S$ and $T$, the function $q(S, T)$ denotes the number of components $C$ of $G \setminus S \setminus T$ such that $f(C) + e(T, C)$ is odd.

**Theorem 9** (Tutte’s $f$-factor theorem). A graph $G$ has a $f$-factor if and only if for all disjoint $S, T \subseteq V(G)$, we have

$$q(S, T) + f(T) - \sum_{v \in T} \deg_{G \setminus S}(v) \leq f(S).$$

We are looking for a $k$-factor, so we set $f(v) = k$ for all $v \in V(G)$. In that case, Tutte’s theorem becomes

**Corollary 10.** A graph $G$ has a $k$-factor if and only if for all disjoint $S, T \subseteq V(G)$, we have

$$q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \leq k|S|.$$ 

Furthermore, Tutte [14] showed Proposition [11] which we will use.

**Proposition 11.** For any graph $G$ on $n$ vertices, if $kn$ is even, then $q(S, T) + k|T| - \sum_{v \in T} \deg_{G \setminus S}(v) \equiv k|S| \pmod{2}$.

We will also use a lemma of Katerinis [8], which we reprove for completeness. Here, for any subset of vertices $X$ in a graph, $c(X)$ denotes the number of connected components of the subgraph induced by $X$.

---

\[1\] In order to clarify the terminology, we observe that a $k$-factor is the same as an $f$-factor if $f$ is the constant function equal to $k$. Whether the prefix of “factor” is a function or an integer will be clear from the context.
Lemma 12 (Katerinis [8]). If $G$ is a Hamiltonian graph and $X \subseteq V(G)$, then $c(G - X) \leq |X|$.

Proof. Let $H$ be a Hamiltonian cycle of $G$. One can verify that $c(H - X) \leq |X|$, since removing the first vertex leaves one component, and any additional vertices removed creates at most one new component. Since $H$ is a subgraph of $G$, we have $c(G - X) \leq |X|$.

All graphs considered in this note are Hamiltonian because we require that the minimum degree is at least $n/2$. Hamiltonicity is thus a direct result of Dirac’s theorem.

Finally, we need the following lemma to verify that Theorem 2 is not claiming that the graph has a $k$-factor for $k > \delta$, and furthermore that when $k = \delta$ the condition $k < \rho$ forces $G$ to have a $k$-factor.

Lemma 13. Let $G, n, \delta, \rho$ be as in Theorem 2. If $k$ is an integer such that $kn$ is even and $\delta \leq k < \rho$, then $k = \delta$ and $G$ has a $k$-factor.

Proof. If we write $k = \delta + t$ for some nonnegative integer $t$, we see

\[
\delta + t < \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2}
\]

\[
\delta^2 + 4\delta t + 4t^2 < 2\delta n - n^2 + 8
\]

\[
n^2 - 2\delta n + \delta^2 + 4\delta t + 4t^2 < 8
\]

\[
(n - \delta)^2 + 4\delta t + 4t^2 < 8
\]

Since $\delta \geq 1$, we have a contradiction if $t > 0$. Hence $t = 0$ which gives $k = \delta$. The calculation above gives $(n - \delta)^2 < 8$, which means $\delta = n - 1$ or $\delta = n - 2$. If $\delta = n - 1 = k$, then $G$ itself is a $k$-factor. If $\delta = n - 2 = k$, then $kn$ being even implies that both $k$ and $n$ are even. In this case, $G$ is a complete graph minus a partial matching. But since $n$ is even, we can extend this partial matching to a full matching. What remains of $G$ outside this matching is a $k$-factor.

4 Proof of Theorem 2

Fix a $k > 0$ such that $kn$ is even and $k < \rho$. Suppose $G$ does not contain a $k$-factor. Then by Tutte’s $f$-factor theorem, for $f$ identically $k$, there exist disjoint $S, T \subseteq V(G)$ such that

\[
q(S, T) + k|T| - \sum_{v \in T} \deg_G(v) > k|S|.
\]

(1)

Since $kn$ is even, the left side has the same parity as the right side by Proposition [11] and we have

\[
q(S, T) + k|T| - \sum_{v \in T} \deg_G(v) > k|S| + 2.
\]

(2)
Set $W = G \setminus (S \cup T)$. Notice that we have $q(S, T) \leq c(W)$. Therefore, we can further reduce (2) to
\[ c(W) + k|T| - \sum_{v \in T} \deg_{G\setminus S}(v) \geq k|S| + 2. \quad (3) \]

We break the proof into 4 cases, establishing a contradiction in each.

**Case 1:** $T = \emptyset$.
We have $|T| = 0$, then $W = G \setminus S$ and $\sum_{v \in T} \deg_{G\setminus S}(v) = 0$, so (3) becomes $c(G \setminus S) \geq k|S| + 2$. Since $G$ is Hamiltonian by Dirac’s theorem, we can apply Lemma 12 to see $c(G - S) \leq |S|$. However, this contradicts $c(G \setminus S) \geq k|S| + 2$ since $k \geq 1$. This concludes Case 1.

Set $\mu = \min_{x \in T}\{\deg_{G\setminus S}(x)\}$. We have $\mu|T| \leq \sum_{v \in T} \deg_{G\setminus S}(v)$, and therefore $k|T| - \sum_{v \in T} \deg_{G\setminus S}(v) \leq (k - \mu)|T|$. Thus (3) becomes
\[ c(W) + (k - \mu)|T| \geq k|S| + 2. \quad (4) \]

We now have three cases based on where the value of $\mu$ falls relative to $k$.

**Case 2:** $T \neq \emptyset$, $\mu \geq k + 1$.
By (4), we have $c(W) - |T| \geq k|S| + 2$. By Lemma 12, we have $c(W) = c(G - (S \cup T)) \leq |S \cup T|$. Thus, we get $|S \cup T| - |T| \geq k|S| + 2$, which becomes $|S| \geq k|S| + 2$, which contradicts $k \geq 1$.

**Case 3:** $T \neq \emptyset$, $\mu = k$.
From (4), we have $c(W) \geq k|S| + 2$.
We use the trivial bound $c(W) \leq |W|$. Furthermore, let us bound $|W| = n - |S| - |T| \leq n - |S| - 1$. Thus (4) becomes
\[ n - |S| - 1 \geq k|S| + 2. \quad (5) \]

Note that a vertex $x \in T$ such that $\mu = \deg_{G\setminus S}(x)$ has at most $|S| + \mu$ neighbors in $G$, and we therefore have $|S| + \mu \geq \delta$ or $|S| \geq \delta - \mu$. From (5) we have
\[ n - 1 - \delta + k \geq k\delta - k^2 + 2, \]
which gives
\[ k^2 - \delta k + n - \delta - 3 \geq 0. \quad (6) \]
Solving (6) using the quadratic formula yields either
\[ k \geq \frac{\delta + \sqrt{\delta^2 - 4(n - \delta - 3)}}{2} \]
We can verify the second solution is less than 1 since substituting \( k = 1 \) into (6) yields a negative value. Consider the first solution. We may assume \( k \leq \delta - 1 \) because of Lemma 13. Therefore, we have

\[
\delta - 1 \geq \frac{\delta + \sqrt{\delta^2 - 4(n - \delta - 3)}}{2} \geq \frac{\delta^2 - 4n + 4\delta + 12}{\delta^2 - 4\delta + 12} \geq \frac{n \geq 2\delta + 2}{2},
\]

which contradicts \( \delta \geq n/2 \).

**Case 4:** \( T \neq \emptyset, 0 \leq \mu \leq k - 1 \).

From (4), we have

\[
c(W) + (k - \mu)|T| \geq k|S| + 2.
\]

We replace \(|T|\) with \( n - |S| - |W| \). Again, we use the trivial bound \( c(W) \leq |W| \). Thus we have

\[
|W| + (k - \mu)(n - |S| - |W|) \geq k|S| + 2.
\]

Recall \(|S| \geq \delta - k \). Therefore,

\[
|W| + (k - \mu)(n - (\delta - \mu) - |W|) \geq (k(\delta - \mu) + 2
\]

\[
(k - \mu)(n + \mu - \delta) + k(\mu - \delta) - 2 \geq (k - 1 - \mu)|W| \geq 0
\]

\[
-\mu^2 + (\delta + 2k - n)\mu + (nk - 2\delta k - 2) \geq 0
\]

(7)

Consider (7) as a quadratic inequality in the variable \( \mu \). Since \( 0 \leq \mu \), the left-hand side achieves its maximum at \( \mu^* = \max\{0, (\delta + 2k - n)/2\} \). Thus, we have a contradiction unless (7) is satisfied when \( \mu = \mu^* \).

If \( \mu^* = 0 \), then (7) reduces to \( k(n - 2\delta) \geq 2 \), which contradicts \( n - 2\delta \leq 0 \). If \( \mu^* = (\delta + 2k - n)/2 \) and \( \mu^* \geq 0 \), then

\[
k \geq \frac{n - \delta}{2}.
\]

(8)

Setting \( \mu = \mu^* \) in (7) gives

\[
(\delta + 2k - n)^2/4 + (nk - 2\delta k - 2) \geq 0
\]

\[
k^2 - k\delta + \delta^2/4 - \delta n/2 + n^2/4 - 2 \geq 0
\]

Solving this quadratic, we obtain roots

\[
\frac{\delta \pm \sqrt{\delta^2 + 2\delta n - \delta^2 - n^2 + 8}}{2}
\]
This gives either
\[ k \geq \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2}, \]
or
\[ k \leq \frac{\delta - \sqrt{2\delta n - n^2 + 8}}{2}. \]  
(9)

In the former, we have a contradiction to our assumption \( k < \frac{\delta + \sqrt{2\delta n - n^2 + 8}}{2} \). In the latter, we have a contradiction, because by combining (9) and (8), we have
\[
\frac{\delta - \sqrt{2\delta n - n^2 + 8}}{2} \geq \frac{n - \delta}{2}
\]
\[
\delta - \sqrt{2\delta n - n^2 + 8} \geq n - \delta
\]
\[
2\delta - n \geq \sqrt{2\delta n - n^2 + 8}
\]
\[
4\delta^2 - 4n\delta + n^2 \geq 2\delta n - n^2 + 8
\]
\[
4\delta^2 - 6n\delta + 2n^2 \geq 8
\]
\[
(n - 2\delta)(n - \delta) \geq 4.
\]
This is impossible, however, since \( n - 2\delta \leq 0 \) and \( n - \delta \) is positive. Thus we have shown that there exists a \( k \)-factor for the largest \( k \) where \( kn \) is even and \( k < \rho \). The smallest this \( k \) can be is \( \rho - 2 \).

This concludes the proof of Theorem 2.

5 Proof of Theorem 3

Construction. Given positive integers such that \( n/2 \leq \delta \leq n - 1 \). Construct \( G \) such that \( V(G) = S \cup T \) with \( S \cap T = \emptyset \).

If \( \delta = n/2 \), then choose \( \sigma = 2 \left\lceil \frac{n - 1}{4} \right\rceil \). If \( \delta > n/2 \), then choose \( \sigma \) to be the largest integer such that \((n - \sigma)(\delta - \sigma)\) is even and
\[
\sigma \leq \frac{n - \sqrt{2\delta n - n^2}}{2}.
\]

Let \( G[S] \) be an independent set of order \( \sigma \), \( G[T] \) be a \((\delta - \sigma)\)-regular graph of order \( n - \sigma \) and the bipartite graph \( G[S,T] \) be complete.

For this construction to work, we need \((n - \sigma)(\delta - \sigma)\) to be even, so that \( G[S] \) to have a \((\delta - \sigma)\)-factor.

Let \( H \) be a \( k \)-factor of \( G \) for some \( k \). In \( H \), there are exactly \( k\sigma \) edges leaving \( S \) because \( G[S] \) is an independent set. Thus, some vertex \( v \) in \( T \) must receive at most \( k\sigma/(n - \sigma) \) edges from \( S \). Since \( v \) has only \( \delta - \sigma \) edges within \( T \), we see that, in \( H \),
\[
\deg_H(v) \leq \frac{k\sigma}{n - \sigma} + (\delta - \sigma).
\]
Since \(\deg_H(v) = k\), we have \(\frac{k\sigma}{n-\sigma} + (\delta - \sigma) \geq k\). Solving this inequality for \(k\), we have

\[
k \leq \frac{(n - \sigma)(\delta - \sigma)}{n - 2\sigma}.
\]  

(10)

Note that Theorem 9 can also be used to derive (10).

If \(\delta = n/2\), then set \(\sigma = \lceil (n - 4)/4 \rceil\), and (10) gives

\[
k \leq \frac{(n - \sigma)(n/2 - \sigma)}{n - 2\sigma} = \frac{n - \sigma}{2} = \frac{n}{2} - \left\lfloor \frac{n - 4}{4} \right\rfloor = \left\lfloor \frac{n + 4}{4} \right\rfloor.
\]

Observe that this construction matches Katerinis' bound because when \(\delta = n/2\), and hence \(n\) is even, \(\lceil (n + 4)/4 \rceil = \lceil (n + 5)/4 \rceil\).

If \(\delta > n/2\), then we set \(\sigma = n - \sqrt{2\delta n - n^2} - \epsilon\), where \(\epsilon \in [0, 2)\) is just some small quantity to make \(\sigma\) an integer of the right parity (recall \(\sigma\) needs to be a certain parity in order for \((n - \sigma)(\delta - \sigma)\) to be even). The minimum degree of the construction is in fact \(\delta\), since the vertices in \(T\) were designed to have degree \(\delta\), and the vertices in \(S\) have degree \(n - \sigma\). To see that \(n - \sigma \geq \delta\),

\[
n - \sigma \geq n - \left(\frac{n}{2} - \frac{\sqrt{2\delta n - n^2}}{2}\right)
\]

\[
= \delta + \frac{n - 2\delta}{2} + \frac{\sqrt{2\delta n - n^2}}{2}
\]

\[
= \delta + \frac{\sqrt{2\delta n - n^2}}{2n} \left(n - \sqrt{2\delta n - n^2}\right) \geq \delta.
\]
Now that we have verified the validity of the construction, we can return to (10):

\[
\begin{align*}
k & \leq \frac{(\delta - \sigma)(n - \sigma)}{n - 2\sigma} \\
& = \frac{\delta - (n - \sqrt{2\delta n - n^2} - 2\epsilon)/2}{n - (n - \sqrt{2\delta n - n^2} - 2\epsilon)/2} \\
& = \frac{(2\delta - n + \sqrt{2\delta n - n^2} + 2\epsilon)(n + \sqrt{2\delta n - n^2} + 2\epsilon)}{4\sqrt{2\delta n - n^2} + 8\epsilon} \\
& = \frac{4\delta n - 2n^2 + (2\delta + 4\epsilon)(2\delta n - n^2) + 4\delta \epsilon + 4\epsilon^2}{4\sqrt{2\delta n - n^2} + 8\epsilon} \\
& = \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{\epsilon^2}{\sqrt{2\delta n - n^2} + 2\epsilon} \\
& \leq \frac{\delta + \sqrt{2\delta n - n^2}}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4}
\end{align*}
\]

This concludes the proof of Theorem 3.

6 Proof of Corollary 4

The construction of the graph \( G \) from Theorem 3 requires that

\[
\begin{align*}
k & \leq \frac{\delta}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
& = \rho + \frac{\sqrt{2\delta n - n^2} - \sqrt{2\delta n - n^2} + 8}{2} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
& = \rho + \frac{2(\sqrt{2\delta n - n^2} + \sqrt{2\delta n - n^2} + 8)}{4} + \frac{4}{\sqrt{2\delta n - n^2} + 4} \\
& \leq \rho + \frac{2(\sqrt{2\delta n - n^2} + 8)}{\sqrt{n} + 8} + \frac{4}{\sqrt{2\delta n - n^2} + 8} \\
& \leq \rho + \frac{2}{\sqrt{n} + 8},
\end{align*}
\]

where the last inequality uses the fact that \( \delta > n/2 \). This establishes Corollary 4.

Acknowledgements

We thank Béla Csaba for some helpful conversations.
References


