# INTEGRAL CLOSURES NOTES

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Let $(R, \mathfrak{m})$ be a local, Noetherian ring. Let $I \subseteq R$ be an ideal of $R$ and $M$ be a finitely generated $R$-module. Let us also assume that $\dim M = \dim R = d$. If $I$ is $\mathfrak{m}$-primary, then how does $\lambda\left((M)/(I^n M)\right)$ behave? (Recall that $I$ is $\mathfrak{m}$-primary if there exists $k \in \mathbb{N}$ so that $\mathfrak{m}^k \subseteq I$.) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be defined by $n \mapsto \lambda\left((M)/(I^n M)\right)$.

**Theorem** (Hilbert-Samuel)**.** *There exists a polynomial $H_{I,M}$ so that $H_{I,M}(n) = f(n)$ for $n \gg 0$. Moreover, $\deg(H_{I,M}) = d$.*

It turns out that $H_{I,M}(x) = \frac{e_R(I,M)}{d!} x^d +$lower order terms, which tells us first that the leading coefficient is divisible by $d!$ and the number $e_R(I, M)$ is called the multiplicity of $I$ with respect to $M$. We see the importance of the multiplicity in the following theorem.

**Theorem.** *Let $R$ be equidimensional. Then $e_R(\mathfrak{m}, R) = 1 \iff R$ is regular.*

**Question:** If $I \subseteq R$ and $r \in R$, then what is the relationship between $I^n$ and $(I, r)^n$?

**Theorem.** *Let $(R, \mathfrak{m})$ be local. Let $I, J \subseteq R$ are $\mathfrak{m}$-primary ideals. If $I$ and $J$ have the same integeral closure, then $e_R(I, M) = e_R(J, M)$.*

The converse is also true if $R$ is equidimensional; namely if for every $M$ with $\dim M = d$ has $e_R(I, M) = e_R(J, M)$, then $I$ and $J$ have the same integral closure. (This is a theorem of Rees.)

## 1. Integral Closure

Let $R$ be a ring and $I \subseteq R$ an ideal and let $r \in R$.

**Definition 1.** (1) $r$ is *integral over $I$* if there exists $n \in \mathbb{N}$ and $a_i \in I^i$ for $i = 1, \dots, n$ so taht $r^n + a_1 r^{n-1} + \cdots + a_n = 0$. This equation is called an *integral relation for $r$ in $I$*.
(2) $\overline{I} = \{r \in R : r \text{ is integral over } I\}$ is the *integral closure of $I$*.
(3) $I$ is *integrally closed* if $I = \overline{I}$.

**Remark 1.** It turns out that $\overline{I}$ is an ideal, but this will take us awhile to actually prove.

**Example 1.** Consider $R = k[x, y]$. Let $I = (x^2, y^2)$. Then we see that $xy \in \overline{I}$. Let $f(T) = T^2 + 0T - x^2y^2$ and note that $f(xy) = 0$ is an integral relation for $xy$ in $I$.

**Remark 2.** We'll prove that if $I = (x^d, y^d)$, then $\overline{I} = (x, y)^d$.

**Exercise 1.** Let $R = k[x_1, \dots, x_n]$ for $k$ a field and suppose that $I$ is generated by $n$-forms of degree $d$ such that $\sqrt{I} = \mathfrak{m}$, then $\overline{I} = (x_1, \dots, x_n)^d$.

**Remark 3.** (1) $I \subseteq \overline{I}$.
(2) $\overline{I} \subseteq \sqrt{I}$. Indeed, let $r \in \overline{I}$. Then there exists $r^n + a_1 r^{n-1} + \cdots + a_n = 0$ and so $r^n = -(a_1 r^{n-1} + \cdots + a_n) \in I$.
(3) Let $\phi : R \longrightarrow S$ be a ring extension and $I \subseteq R$ be an ideal, then $\phi(\overline{I}) \subseteq \overline{\phi(I)}$; i.e., $\overline{I}S \subseteq \overline{IS}$. Indeed, if $r \in \overline{I}$, then $r^n + a_1 r^{n-1} + \cdots + a_n = 0$ and applying $\phi$ gives us that $\phi(r)^n + \phi(a_1)\phi(r)^{n-1} + \cdots + \phi(a_n) = 0$.

**Question:** What condition do we need for $\phi$ in the last remark do we need in order to get the equality $\overline{I}S = \overline{IS}$?

first **Proposition 1.** *Let $R$ be a Noetherian ring and $I \subseteq R$ an ideal. Let $N = \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p}$ and set $R' = (R)/(N)$ (and let $\phi = \pi : R \longrightarrow R' = (R)/(N)$). Then the following hold:*
(1) $\overline{I}R' = \overline{IR'}$.

(2) *If $r \in R$, then $r \in \overline{I}$ if and only if the image of $r$ in $(R)/(\mathfrak{p})$ is integral over the image of $I$ in $(R)/(\mathfrak{p})$ for all $\mathfrak{p} \in \min(R)$. (Recall that the image over $I$ in $(R)/(\mathfrak{p})$ is $(I+\mathfrak{p})/(\mathfrak{p}) \subseteq (R)/(\mathfrak{p})$.*

*Proof.* (1) Note that $\overline{I}R' \subseteq \overline{IR'}$. We need to show "$\supseteq$." Let $r \in R'$ be such that $r' \in \overline{IR'}$. Then there exists an integral relation $(r')^n + a_1'(r')^{n-1} + \cdots + a_n' = 0$ for $a_i' \in (IR')^i$. Let $r \in R$ be such that $r \equiv r' \pmod N$ and $a_i \equiv a_i' \pmod N$. Then we get $r^n + a_1 r^{n-1} + \cdots + a_n = m \in N$. Then there exists $s$ so that $N^s = 0$. Then we see that

intrel (1) $$(r^n + \cdots + a_n)^s = 0,$$

and Equation (intrel 1) is an integral relation for $r$, which means that $r \in \overline{I}$ and hence $r' \in \overline{I}R'$.

**Remark 4.** $N \subseteq \overline{I}$ for any $I \subseteq R$.

(2) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minal primes of $R$. Then set $R_i = (R)/(\mathfrak{p}_i)$.
"$\Longrightarrow$": Now if $r \in \overline{I}$, then $r_i \in \overline{I}R_i \subseteq \overline{IR_i}$.
"$\Longleftarrow$": For each $i$, there exists $f_i(T) = T^{n_i} + a_{1,i}T^{n_i-1} + \cdots + a_{n_i,i}$ an integral equation with $a_{j,i} \in (IR_i)^j$ for $j \in \{1, \ldots, n_i\}$. Also, we have that $f_i(r_i) = 0$. Lift $f_i$ to $R$ to get $\widetilde{f_i} = T^{n_i} + a_{1,i}'T^{n_i-1} + \cdots + a_{n_i,i}'$, where $a_{j,i}' \equiv a_{j,i} \pmod{)}p_i$. Thus, $\widetilde{f_i}(r) = p_i \in \mathfrak{p}_i$. Then we see that

$$\prod_{i=1}^n \widetilde{f_i}(r) = \prod_{i=1}^n p_i \in \mathfrak{p}_1 \cdots \mathfrak{p}_n \subseteq \bigcap_{i=1}^n \mathfrak{p}_i = N.$$

Then there exists an $s$ so that $\left(\prod_{i=1}^n f_i(r)\right)^s \in N^s = 0$ and so $r \in \overline{I}$.

□

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**Exercise 2.** Let $R$ be a UFD. If $I = (f)$, then $I = \overline{I}$.

stepdown **Proposition 2.** *Suppose $R$ is Noetherian and $I \subseteq R$ and $r \in R$. The following are equivalent:*

(1) $r \in \overline{I}$
(2) *there exists a finitely generated $R$-module $M$ so that $rM \subseteq IM$ and if $aM = 0$ for some $a \in R$, then $ar \in \sqrt{0}$.*

*If $I$ contains a non-zero divisor, then*

($2'$) *there exists a finitely generated faithful $R$-module $M$ so that $rM \subseteq IM$.*

Recall that an $R$-module is faithful if $aM = 0 \Longrightarrow a = 0$.

**Exercise 3.** (1) If $R \xrightarrow{\phi} S$ is finite and $I \subseteq R$, then $\overline{I} = \overline{IS} \cap R$.

(2) If $R \xrightarrow{\phi} S$ is faithfully flat, then $\overline{I} = \overline{IS} \cap R$. (Here, you may assume that $I \subseteq R$ and also that $\overline{I}$ is an ideal.)

**Example 2.** Let $k$ be a field with $\operatorname{char} k = 0$. Suppose that $I \subseteq k[x,y]$ is given by $I = (x^2+y^3, x^2y, x^3) = (x^2+y^3, \mathfrak{m}^4)$, where $\mathfrak{m} = (x,y)$. We'll show that $\overline{I} = I$. Consider

$$R = k[x,y] \longrightarrow S = \frac{k[x,y,z]}{(x-z^3, y-z^2)} \cong k[z]$$

Let $f \in \overline{I}$. Then we see that $f \in \overline{I}S$ and so $f \in \overline{IS}$ by persistence. Notice that $IS$ is

$$\begin{aligned} IS &= \left((z^3)^2 + (z^2)^3, (z^3)^2 \cdot z^2, z^9\right) \\ &= (z^6). \end{aligned}$$

By one of the exercises, $\overline{IS} = IS$. Thus, we have $\overline{I} = \overline{IS} \cap R = IS \cap R = (z^6, x-z^3, y-z^2) \cap k[x,y] = I$ (the last equality is by M2).

dettrick **Lemma 3** (Determinant Trick). *Let $M$ be a finitely generated $R$-module and $I \subseteq R$ an ideal. Let $\phi \in \mathrm{Hom}_R(M,M)$. If $\phi(M) \subseteq IM$, then there exists an equation*

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$$

*for $a_i \in I^i$ for $i \in \{1,\dots,n\}$.*

*Proof.* Let $m_1,\dots,m_n$ be a set of generators for $M$. Then we have

$$\begin{aligned}\phi(m_1) &= a_{1,1}m_1 + \cdots + a_{1,n}m_n\\ \vdots\ \ &\ \ \vdots\\ \phi(m_n) &= a_{n,1}m_1 + \cdots + a_{n,n}m_n.\end{aligned}$$

Define the matrix $A$ by

$$A = \begin{pmatrix} a_{1,1}-\phi & a_{1,2} & \dots & a_{1,n}\\ a_{2,1} & a_{2,2}-\phi & \dots & a_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n,1} & a_{n,2} & \dots & a_{n,n}\end{pmatrix}.$$

Then $\mathrm{adj}(A)\cdot A \begin{bmatrix} m_1\\ \vdots\\ m_n\end{bmatrix} = 0$ implies that $\det(A)\mathrm{Id}_n \begin{bmatrix} m_1\\ \vdots\\ m_n\end{bmatrix} = 0$, which means that $\det(A)M = 0$. This equation is actually just the equation we are looking for. □

**Lemma 4.** *Let $R$ be Noetherian and $I \subseteq R$. The following are equivalent:*

(1) $r \in \overline{I}$
(2) *there exists an $n$ so that $(I+r)^n = I(I+r)^{n-1}$.*

*Proof.*

$$\begin{aligned} r \in \overline{I} &\iff \exists r^n + a_1r^{n-1} + \cdots + a_n = 0\\ &\iff r^n = -(a_1r^{n-1} + \cdots + a_n)\\ &\iff r^n \in Ir^{n-1} + \cdots + I^n = I(r^{n-1} + \cdots + I^{n-1})\\ &\qquad\qquad\qquad\qquad\qquad\qquad = I(I+r)^{n-1}.\end{aligned}$$

□

*Proof.* (of Proposition stepdown 2)
$1 \Longrightarrow 2$: If $r \in \overline{I}$, then there exists an equation $r^n + a_1r^{n-1} + \cdots + a_n = 0$. Let $M = (I+r)^{n-1}$. Then $rM = r(I+r)^{n-1} \subseteq (I+r)^n = I(I+r)^{n-1} = IM$. If $aM = 0$, then $a(I+r)^{n-1} = 0$ and so $ar^{n-1} = 0$, which means that $(ar)^{n-1} = 0$.

Now assume that $I$ has a non-zero divisor, say $x$. This also means that $x^{n-1}$ is a non-zero divisor. If $a(I+r)^{n-1} = 0$, then $ax^{n-1} = 0$, which means that $a = 0$.

$2 \Longrightarrow 1$: Let $rM \subseteq IM$. By Lemma dettrick 3, there exists

$$(r^n + a_1r^{n-1} + \cdots + a_n) = b,$$

where $bM = 0$. Thus, $rb \in \sqrt{0}$ and so there exists an $s$ so that $(rb)^s = 0$. Note that $(rb)^s = 0$ is an integral equation with degree $(n+1)s$. □

## 2. What can the "Determinant Trick" do for you?

**Theorem.** *Let $(R,\mathfrak{m})$ be local and suppose that $M$ is a finitely generated $R$-module. Suppose that $I \subseteq R$ is $\mathfrak{m}$-primary such that $I = \overline{I}$. If $Tor_k^R(M,(R)/(I)) = 0$, then $pd_R M < k$.*

**Theorem** (Iyengar,?). *Let $I = \mathfrak{m}$. If $Tor_k^R(M,(R)/(\mathfrak{m}^n)) = 0$ and depth $R > 0$, then $pd_R M < k$.*

**Definition 2.** An ideal $I$ is *normal* if $\overline{I^n} = I^n$ for all $n \geq 1$.

*Proof.* Suppose $\dim R \geq 1$. Let

$$F_{k+1} \xrightarrow{\partial_{k+1}} F_k \xrightarrow{\partial_k} F_{k-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution. Denote by $-'$ the functor $- \otimes (R)/(I)$.

I claim that $\mathrm{Im}\partial'_k \cap \mathrm{soc}(F'_{k-1}) = 0$. Assume the claim and we'll prove the theorem. This tells us that $\mathrm{Im}\partial'_k = 0$, which means that $\ker \partial'_k = (F_k)/(IF_k)$. Then note that

$$Tor_k^R(M, (R)/(I)) = \frac{\ker \partial'_k}{\mathrm{Im}\partial'_{k+1}} = \frac{F_k}{\mathrm{Im}\partial_{k+1} + IF_k}.$$

This means that $F_k \subseteq \mathrm{Im}\partial_{k+1} + IF_k \subseteq \mathfrak{m}F_k$, which implies that $F_k = 0$ by NAK. $\square$

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Remember that last time we were in the process of proving the following claim.

intiszero **Claim 4.1.**

$$\mathit{image}(\partial'_k) \cap \mathit{socle}(F'_{k-1}) = 0.$$

Recall that if $M$ is an $R$-module, then $\mathrm{socle}(M) = \{x \in M : \mathfrak{m}x = 0\}$.

*Proof.* (of Claim 4.1 intiszero) Let $x' \in \mathrm{image}(\partial'_k) \cap \mathrm{socle}(F'_{k-1})$. Then there exists $y' \in F'_k$ so that $x' = \partial'_k(y')$. For all $a \in \mathfrak{m}R'$, note that $0 = ax' = \partial'_k(ay')$. Thus, $ay' \in \ker \partial'_k$. However, $Tor_k(M, (R)/(I)) = \frac{\ker \partial'_k}{\mathrm{im}\partial'_{k+1}} = 0$. Thus, $ay' \in \mathrm{im}\partial'_{k+1}$. Therefore, we know that there must exist a $z'$ so that $\partial'_{k+1}(z') = ay'$. If we lift to $R$, we find that $x = \partial_k(y) + w$ for $w \in IF_{k-1}$. Also, $ay = \partial_{k+1}(z) + w_1$ for $w_1 \in IF_k$. In particular, $ax = \partial_k(ay) + aw = \partial_k(\partial_{k+1}(z)) + \partial_k(w_1) + aw = 0 + 0 + w_1$. Note that $ax \in \mathfrak{m}IF_{k-1} + \mathfrak{m}IF_{k-1}$. $(x_1, \ldots, x_e) = \mathfrak{m}_x \subset \mathfrak{m}IF_{k-1}$. Thus, $\mathfrak{m}_{x_i} \subseteq \mathfrak{m}I$ and hence $x_i\mathfrak{m} \subseteq I\mathfrak{m}$. By the determinant trick (as long as we have that if $a\mathfrak{m} = 0$, then $(ax_i)^r = 0$), $x_i \in \overline{I} = I$. $\square$

**Exercise 4.** Let $(R, \mathfrak{m})$ be a local ring. Let $a \in \mathfrak{m}$. If $a\mathfrak{m} = 0$, then $a \in \sqrt{0}$.

*Proof.* Since $a \in \mathfrak{m}$ and $a\mathfrak{m} = 0$, we see that $a^2 = 0$ and so $a \in \sqrt{0}$. $\square$

## 3. Integral Closure of Rings

**Definition 3.** Let $R \subseteq S$ be a ring extension. An element $x \in S$ is *integral over* $R$ if there exists an equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_i \in R$.

**Remark 5.** If $r \in R$, then $r$ is integral over $R$. Also, $R$ is integrally closed over $S$ if there is no $r \in S \setminus R$ so that $r$ is integral over $R$.

**Example 3.** $\mathbb{Z}$ is integrally closed over $\mathbb{Q}$. Note that if we consider $\mathbb{Z} \subseteq \mathbb{C}$, then $i^2 - 1 = 0$ and so $i$ is integral over $\mathbb{Z}$.

The integral closure of $R$ is often the integral closure of $R$ in its quotient field (when $R$ is a domain).

**Remark 6.** Every UFD is integrally closed. (Proof: cf. the book.)

This implies that $\mathbb{Z}$ is integrally closed and also that every regular ring is integrally closed.

**Exercise 5.** The integral closure of $k[t^2, t^3]$ is $k[t]$.

**Exercise 6.** Let $R \subseteq S$ is an integral extension and let $\mathfrak{q} \in \mathrm{Spec}(S)$ and set $\mathfrak{p} = \mathfrak{q} \cap R$ and let $W$ be a multiplicatively closed subset of $R$. Then the following hold:

(1) $(R)/(\mathfrak{p}) \subseteq (S)/(\mathfrak{q})$ is still an integral extension.
(2) $W^{-1}R \subseteq W^{-1}S$ is an integral extension.

**Lemma 5.** *Let $R \subseteq S$ and suppose that $x_1, \ldots, x_n \in S$. The following are equivalent:*

(1) *$x_i$ is integral over $R$ for each $i = 1, \ldots, n$,*
(2) *$R[x_1, \ldots, x_n]$ is a finitely generated $R$-module,*
(3) *there exists a faithful finitely generated $R$-module $M \subseteq R[x_1, \ldots, x_n]$ so that $x_iM \subseteq M$.*

*Proof.* If $R \subseteq S \subseteq T$ is a ring extension, and $t$ is integral over $R$, then the exact same equation shows that $t$ is integral over $S$. Without loss of generality, we may reduce the lemma to the case where $n = 1$, and set $x := x_1$.

$1 \implies 2$: Since $x$ is integral, there exists $x^n + a_1 x^{n-1} + \cdots + a_n = 0$. Thus, $x^n = -a_1 x^{n-1} - \cdots - a_n$. This means that $R[x] = R \cdot 1 + R \cdot x + \cdots + R \cdot x^{n-1}$.

$2 \implies 3$: Take $R[x] = M$ and then $xR[x] \subseteq R[x]$.

$3 \implies 1$: By the determinant trick, where $\phi$ is multiplication by $x$ and $I = R$, then there exists an equation $x^n + a_1 x^{n-1} + \cdots + a_n = b$, where $bM = 0$. Since $M$ is faithful, we see that $b = 0$, which means that we really have an equation of integral dependence. □

**Corollary 6.** *The integral closure of $R \subseteq S$ is a ring.*

*Proof.* If $x, y \in S$ are integral over $R$, then by part 2 in the previous lemma, $R[x, y]$ is a finitely generated $R$-module. However, $R[x, y, xy] = R[x, y, x - y] = R[x, y]$ and since $2 \implies 1$, we see that $x - y$ and $xy$ are integral over $R$. □

**Lemma 7.** *Let $R \subseteq S$ be an integral extension of domains. $R$ is a field if and only if $S$ is a field.*

*Proof.* $\Longleftarrow$: Suppose that $S$ is a field. Pick $x \in R$. In $S$ there exists $x^{-1} \in S$. There exists an integral equation $(x^{-1})^n + a_1 (x^{-1})^{n-1} + \cdots + a_n = 0$ for $a_i \in R$ for $i = 1, \ldots, n$. Multiplying the equation by $x^n$ to get $1 = a_1 x + \cdots + a_n x^n = 1 + x(a_1 + \cdots + a_n x^{n-1}) = 0$. Then $(a_1 + \cdots + a_n x^{n-1}) \in R$ is an inverse of $x$.

$\Longrightarrow$: Let $x \in S$. There exists an integral equation $x^n + a_1 x^{n-1} + \cdots + a_1 = 0$ for $a_i \in R$ for $i = 1, \ldots, n$. Thus, we have $x(x^{n-1} + \ldots a_{n-1}) = -a_n \in R$. Thus, we have $x^{-1} = (x^{n-1} + \cdots + a_{n-1})(-a_n)^{-1}$. □

primes **Corollary 8.** *If $R \subseteq S$ is integral and $\mathfrak{q} \in \operatorname{Spec}(S)$, and $\mathfrak{p} := \mathfrak{q} \cap R$, then we know that $\mathfrak{p}$ is prime. Furthermore, $\mathfrak{p}$ is maximal if and only if $\mathfrak{q}$ is maximal.*

*Proof.* Note that $(R)/(\mathfrak{p}) \subseteq (S)/(\mathfrak{q})$ and individually, one is a field if and only if the other is a field. □

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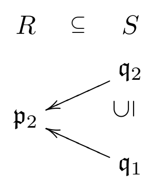
Our goal is to show that if $R \subseteq S$ is an integral extension, then $\dim R = \dim S$.

**Theorem 9** (Incomparability)**.** *Let $R \subseteq S$ be an integral extension. Suppose that $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}(S)$. If $\mathfrak{p}_1 := \mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R =: \mathfrak{p}_2$, then $\mathfrak{q}_1 = \mathfrak{q}_2$.*

*Proof.* Replace $R$ with $R_{\mathfrak{p}_2}$ and $S$ with $S_{R \setminus \mathfrak{p}_2}$, and note that $R \subseteq S$ is an integral extension. It is enough to show the theorem when $\mathfrak{q}_2$ contracts to a maximal ideal. We have



By corollary 8, we see that $\mathfrak{q}_i$ both being maximal implies that $\mathfrak{q}_1 = \mathfrak{q}_2$. □

Note that if $\mathfrak{p}_d \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0$, then $\dim S \leq \dim R$ since there exists a prime that lies over each of these primes by the following theorem.

**Theorem 10** (Lying Over)**.** *Let $R \subseteq S$ be integral. If $\mathfrak{p} \in \operatorname{Spec}(R)$, then there exists $\mathfrak{q} \in \operatorname{Spec}(S)$ so that $\mathfrak{q} \cap R = \mathfrak{p}$.*

*Proof.* Replace $R$ by $R_{\mathfrak{p}}$ and $S$ by $S_{R \setminus \mathfrak{p}}$. It is enough to show the theorem where $R$ is local and $\mathfrak{p}$ is the maximal ideal. Pick any $\mathfrak{q}$ maximal in $S$. Look at $\mathfrak{q} \cap R =: \mathfrak{p}'$. By corollary 8, $\mathfrak{p}'$ is maximal and so it must be $\mathfrak{p}$. □

**Theorem 11** (Going Up)**.** *Let $R \subseteq S$ be integral. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \in \operatorname{Spec}(R)$ and $\mathfrak{q}_1 \in \operatorname{Spec}(S)$ such that $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$. There exists $\mathfrak{q}_2 \in \operatorname{Spec}(S)$ with $\mathfrak{q}_2 \supseteq \mathfrak{q}_1$ so that $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$.*

*Proof.* Replace $R \subseteq S$ by $R_{\mathfrak{p}_2} \subseteq S_{R \setminus \mathfrak{p}_2}$and then in turn by $R' := \frac{R_{\mathfrak{p}_2}}{\mathfrak{p}_1 R_{\mathfrak{p}_2}} \subseteq \frac{S_{R \setminus \mathfrak{p}_2}}{\mathfrak{q}_1 S_{R \setminus \mathfrak{p}_2}} =: S'$. By lying over, there exists an ideal in $S'$ that contracts back to $\mathfrak{p}_2 R'$. □

$R \subseteq S$ is integral, then $\dim R = \dim S$. I claim that if we let $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} := \mathfrak{q} \cap R$, then $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p}$.

*False Proof.* Note that $R_\mathfrak{p} \hookrightarrow S_\mathfrak{q}$ is integral. Then $\operatorname{ht}\mathfrak{p} = \dim R_\mathfrak{p} = \dim S_\mathfrak{q} = \operatorname{ht}\mathfrak{q}$.
The problem is that actually we ought to have $R_\mathfrak{p} \hookrightarrow S_{R \setminus \mathfrak{p}}$. □

**Example 4.** Let $R$ be a domain and $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\operatorname{ht}\mathfrak{p} > 0$. Define $x \mapsto (x, \bar{x})$, where $R \longrightarrow R \oplus (R)/(\mathfrak{p})$. Let $\mathfrak{q} = (R, 0)$ and note that $\mathfrak{q} \longrightarrow \mathfrak{p}$, but $\operatorname{ht}(\mathfrak{q}) = 0$.

Let $R$ be a domain and set $K = \mathbb{Q}(R)$.

**Lemma 12.** *Let $I \subseteq (K)$ and $J \subseteq (K)R$ be $R$-modules. Then any $\phi \in hom_R(I, J)$ is the multiplication by an element in $K$.*

*Proof.* Let $\phi \in \hom_R(I, J)$. Let $x, y \in I$ and note that $\phi(xy) = x\phi(y) = y\phi(x)$ and $\phi(y) = \frac{\phi(x)}{x} y$, where $\frac{\phi(x)}{x} \in K$. Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then $bd\phi(\frac{a}{b}\frac{c}{d}) = \phi(ac) = a\phi(c) = c\phi(a)$. Note that $\phi(a) = \phi(\frac{a}{b}b) = b\phi(\frac{a}{b})$ and so $\phi(\frac{a}{b}) = \frac{\phi(a)}{b}$. □

We have $(J :_K I) \xrightarrow{\cong} \hom_R(I, J)$. Our goal now is to show that if $R$ is a Noetherian domain, then $\overline{R} = \bigcup_{\substack{I \neq 0 \\ \text{fractional} \\ \text{ideal.}}} \hom_R(I, I)$.

**Definition 4.** An ideal $I \subseteq K$ is a *fractional ideal* if there exists $a \in K$ so that $aI \subseteq R$.

If $R$ is Noetherian, and $I \subseteq K$, then $I$ is fractional if and only if $I$ is a finitely generated $R$-module.

**Definition 5.** If $I$ is fractional, then $I^{-1} = \{x \in K : xI \subseteq R\}$.

**Example 5.** Let $R$ be Noetherian and $\hat{R}$ reduced. Suppose that $R$ is one-dimensional. Then $R$ is Gorenstein if and only if $\mathfrak{m}^{-1}$ is 2-generated.

*Proof.* "$\subseteq$" $\overline{R} \supseteq \hom_R(I, I)$ and let $\phi \in \hom_R(I, I)$. Then $\phi = \cdot r$ for $r \in K$. Thus, $rI \subseteq I$ and so $r \in \overline{R}$ by the determinant trick. Let $r \in \overline{R}$. Then $R \subseteq R[r]$ and $R[r]$ is a finitely generated $R$-module. Set $J = R :_R R[r]$. Thus, $rJ \subseteq R$, but it is more: $rJ \subseteq J \Longrightarrow r \in \operatorname{Hom}(J, J)$. Now $rJ \subseteq J$ and so $rJr \subseteq R$, but $Jr^2 = rJr$ and $r^2 \in R[r]$. □

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**Definition 6.** Let $R \subseteq \overline{R} \subseteq \mathbb{Q}(R) =: K$, where $R$ is reduced. The *conductor* is $C_R := (R :_K \overline{R})$.

**Exercise 7.**
(1) $C_R$ is an ideal of $R$.
(2) $C_R$ is the largest common ideal for $R$ and $\overline{R}$.
(3) If $R$ is Noetherian, then $R \subseteq \overline{R}$ is a finitely generated extension if and only if $C_R$ contains a non-zero divisor.

**Theorem.** *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring with $|k| = \infty$. Let $\overline{R}$ be a finitely generated $R$-module and suppose that $w_R \subseteq R$ (the canonical module) exists. Then $R$ is Gorenstein if and only if $\ell((\overline{R})/(C_R)) = 2\ell((R)/(C_R))$.*

**Theorem 13.** *Let $R \subseteq S$ be $\mathbb{Z}^m$-graded rings. The integral closure of $R$ in $S$ is $\mathbb{Z}^m$-graded.*

**Example 6.** Consider $R = k[x, y, z]$. Then $R$ is $\mathbb{Z}$-graded, where

$$R_i = \begin{cases} 0, & i < 0 \\ k, & i = 0 \\ (x^a y^b z^{i-a-b}), & a, b, i - a - b \geq 0. \end{cases}$$

This also is $\mathbb{Z}^3$-graded, where (for $a, b, c \geq 0$)

$$R_{(a,b,c)} = kx^a y^b z^c.$$

Finally, note further that $R$ is even $\mathbb{Z}^2$-graded! This is where we have (for $a, b \geq 0$)

$$R_{(a,b)} = k[z]x^a y^b.$$

*Proof.* We will proceed via induction on $n$. Assume that $n = 1$ is done. By induction, the theorem holds up through $n-1$. Let $s \in S$ be integral over $R$. Note that we may decompose $s$ as $s = \sum_{a \in \Lambda \subseteq \mathbb{Z}^n} s_a$. We need to show that $s_a$ is integral over $R$ for each $a \in \Lambda$. The homogeneous element of $s$ in the $\mathbb{Z}^{n-1}$-grading is in the integral closure. Thus we get for each $u \in \mathbb{Z}^{n-1}$:

$$s_u = \sum_{(u,x) \in \Lambda \subseteq \mathbb{Z}^{n-1} \times \mathbb{Z}} s_{(u,x)}.$$

We can give $R \subseteq S$ a $\mathbb{Z}$-grading by just looking at only the last component. Thus, we see that each $s_u$ is in the integral closure of $R$ in $S$ by induction.

Now let us do the base case, $n = 1$. Let $s = s_k + \cdots + s_{k+m}$ that is integral over $R$. Assume that there are $m+1$ units $\alpha_i$ for $i = 1, \ldots, m+1$ in $R_0$, different from 1, such that $\alpha_i \neq \alpha_j$ and $\alpha_i - \alpha_j$ is a unit when $i < j$. For each $i = 1, \ldots, m+1$, define $\sigma_i : S \longrightarrow S$ given by $f \mapsto \alpha_i^d f$ (for $f$ homogeneous of degree $d$). If $T$ is the integral closure of $R$ in $S$, then $\sigma_i(T) = T$. Then $\sigma_i(s) = \alpha_i^k s_k + \alpha_i^{k+1} s_{k+1} + \cdots + \alpha_i^{k+m} s_{k+m} = \alpha_i^k(s_k + \alpha_i s_{k+1} + \cdots + \alpha_i^m s_{k+m})$. Note that $(s_k + \alpha_i s_{k+1} + \cdots + \alpha_i^m s_{k+m}) \in T$. Define the matrix $A$ by

$$A = \begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_1^m \\ 1 & \alpha_2 & \ldots & \alpha_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \ldots & \alpha_m^m \end{pmatrix}.$$

Notice that

$$A \underbrace{\begin{pmatrix} s_k \\ s_{k+1} \\ \vdots \\ s_{k+m} \end{pmatrix}}_{=:\vec{v}} \in T^{n+1}.$$

We see that $\det(A) = \prod_{i<j}(\alpha_i - \alpha_j)$. Then we see that $\operatorname{adj}(A)A\vec{v} \in T^{n+1}$. This means that $(\det(A))s_{k+i} \in T$ for $i = 0, \ldots, m$ and so $s_{k+i} \in T$.

Finally, we need to show that the assumption on the $m+1$ units with $\alpha_i - \alpha_j$ being a unit for $i \neq j$ and $i < j$ holds. Suppose, toward a contradiction, that this doesn't hold for $R \subseteq S$. Note that

$$R \subseteq R' := R[t_i, t_i^{-1}, (t_i - t_j)^{-1} : i = 1, \ldots, m+1, \text{ and } i < j].$$

Furthermore,

$$R' \subseteq S' := S[t_i, t_i^{-1}, (t_i - T_j)^{-1} : i = 1, \ldots, m+1, \text{ and } i < j].$$

$S$ is integral over $R$ and hence $R'$. Because of the assumption, each $s_j$ is integral over $R'$. Pick $s_1$ so that $s_1^\ell + f_1 s_1^{\ell-1} + \cdots + f_\ell = 0$ for $f_i \in R'$ for all $i$. Thus, we see that $f_i$ is a function of $t_i$, $t_i^{-1}$ and $(t_i - t_j)^{-1}$. The common denominator of $f_i$ will be $t_1^{a_1} t_2^{a_2} \ldots t_{m+1}^{a_{m+1}} \prod_{i<j}(t_i - t_j)^{a_{i,j}}$. Thus, $b s_1^\ell + \widetilde{f}_1 s_1^{\ell-1} + \cdots + \widetilde{f}_\ell = 0$ for $\widetilde{f}_i \in R[t_1, \ldots, t_{m+1}]$. Take the monomial

$$f \cdot t_1^{a_1 + a_{1,2} + a_{1,3} + \cdots + a_{1,m+1}} t_2^{a_2 + a_{2,3} + \cdots + a_{2,m+1}} \ldots t_{m+1}^{a_{m+1}}.$$

Then we see that $f = s_1^\ell + \widetilde{\widetilde{f}}_1 s_1^{\ell-1} + \cdots + \widetilde{\widetilde{f}}_\ell = 0$ for $\widetilde{\widetilde{f}}_i \in R$. □

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**Definition 7.** If $I \subseteq R$ is an ideal, then the *Rees algebra of* $I$ is $R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots$. The *extended Rees algebra* is $R[It, t^{-1}] = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \ldots$.

**Theorem 14.** *Let $I \subseteq R$ be an ideal. The integral closure of $R[It] \subseteq R[t]$ is*

$$R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \ldots.$$

*Proof.* $R[It] \subseteq R[t]$ is $\mathbb{Z}$-graded. From the last theorem, the integral closure looks like

$$S := R \oplus J_1 t \oplus J_2 t^2 \oplus \ldots,$$

so it is enough to show that $J_n = \overline{I^n}$. Let $st^n \in S$. Then $s \in J_n$. Since $st^n$ is integral, we have an equation

$$(st^n)^d + (st^n)^{d-1} f_1 + \cdots + f_d = 0$$

for $f_i \in R[It]$. Note that if we consider the $t^{nd}$ component, we have

$$t^{nd}(s^d + s^{d-1}a_1 + \cdots + a_d) = 0$$

and $a_i \in I^{ni}$. Thus, we see that

$$s^d + s^{d-1}a_1 + \cdots + a_d = 0$$

is an integral equation for $s \in \overline{I^n}$. If $s \in \overline{I^n}$, then there exists an equation of the form

$$s^d + a_1 s^{d-1} + \cdots + a_d = 0$$

for $a_i \in (I^n)^i$. Multiplying by $t^{nd}$, we get

$$(st^n)^d + a_1 t^n (st^n)^{d-1} + \cdots + a_d t^{nd} = 0$$

for $a_i t^{ni} \in I^{ni} t^{ni} \subseteq R[It]$. □

**Corollary 15.** *$\overline{I}$ is an ideal of $R$.*

*Proof.* Note that $R[It] \subseteq R[t]$ and the integral closure is $S = R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \ldots$. By one of our previous corollaries, we know that $S$ is a ring. If $a, b \in \overline{I}$, then $at + bt \in S \Longrightarrow (a+b)t \in S \Longrightarrow (a+b) \in \overline{I}$. Similarly for multiplication by an element $r$, we see that $\overline{I}$ is closed. □

**Exercise 8.** Prove that $\overline{\overline{I}} = \overline{I}$.

**Corollary 16.** *Let $R$ be $\mathbb{N}^d$-graded. If $I$ is homogeneous, then $\overline{I}$ is homogeneous.*

*Proof.* Note that $R[It]$ is $\mathbb{N}^{d+1}$-graded. If $u \in \overline{I}$, then $u = \sum_{a \in \mathbb{N}^d} u_a$. Note that $ut \in S \Longrightarrow ut = \sum_{a \in \mathbb{N}^d} u_a t$. Since $S$ is graded, $u_a t \in S$ and so $U_a \in \overline{I}$. □

**Exercise 9.** Let $R = k[x_1, \ldots, x_n]$ and let $I$ be a monomial ideal. Prove that $\overline{I}$ is a monomial ideal.

Our goal is to determine what the dimension of $R[It]$ is.

**Definition 8.** The *associated graded ring* is

$$\operatorname{gr}_I R = (R)/(I) \oplus (I)/(I^2) \oplus \cdots = (R[It])/(IR[It]).$$

**Definition 9.** Let $(R, \mathfrak{m})$ be local. The *fiber cone* is defined to be

$$\mathcal{F}_I(R) = (R)/(\mathfrak{m}) \oplus (I)/(\mathfrak{m}I) \oplus (I^2)/(\mathfrak{m}I^2) \oplus \cdots = (R[It])/(\mathfrak{m}R[It]).$$

**Theorem 17.** *Let $I \subseteq R$ be an ideal with $\dim R < \infty$. Then we see that*

$$\dim R[It] = \begin{cases} \dim R, & \text{if } I \subseteq \mathfrak{p} \text{ for all } \mathfrak{p} \in \min(R) \\ \dim R + 1, & \text{otherwise.} \end{cases}$$

**Remark 7.** Every ideal $J \subseteq R$ is contraced from an ideal of $R[It]$.

*Proof.* $J \subseteq JR[It] \cap R \subseteq JR[t] \cap R = J$ and so equality holds along the way. □

**Remark 8.** For every $\mathfrak{p} \in \operatorname{Spec}(R)$, $\Delta(\mathfrak{p}) := \mathfrak{p}R[t] \cap R[It]$ is a prime in $R[It]$.

**Proposition 18.** $\min(R[It]) = \{\Delta(\mathfrak{p}) : \mathfrak{p}$ *is minimal over* $R\}$.

**Exercise 10.** Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Show that $\mathfrak{p} \in \min(R)$ if and only if there exists $s \in R$, non-nilpotent so that $sp^\ell = 0$ for $\ell \gg 0$.

*Proof.* Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be minimal over $R$. Let $\mathfrak{q} \in \min(R[It])$. $\mathcal{N}(R[It]) = \mathcal{N}(R[t]) \cap R[It] = (\bigcap_{i=1}^n \mathfrak{p}_i R[t]) \cap R[It] \subseteq \mathfrak{q}$. Thus, $\bigcap_{i=1}^n (\mathfrak{p}_i R[t] \cap R[It]) = \bigcap_{i=1}^n \Delta(\mathfrak{p}_i)$. Thus, there exists $i$ so that $\mathfrak{q} \supseteq \Delta(\mathfrak{p}_i)$ and hence $\mathfrak{q} = \Delta(\mathfrak{p}_i)$.

"$\supseteq$" Let $\mathfrak{p} \in \min(R)$. Then there exists $s \in R$ that is non-nilpotent and $\ell \in \mathbb{N}$ so that $s\mathfrak{p}^\ell = 0$ and so $s(\mathfrak{p}R[t])^\ell = 0 \Longrightarrow s(\mathfrak{p}R[t] \cap R[It])^\ell = 0 \Longrightarrow \mathfrak{p}R[t] \cap R[It]$ is minimal.

Thus,

$$\begin{aligned}\dim R[It] &= \max\{\dim \frac{R[It]}{\mathfrak{q}} : \mathfrak{q} \in \min(R[It])\} \\ &= \max\{\dim \frac{R[It]}{\Delta(\mathfrak{q})} : \mathfrak{q} \in \min(R)\} \\ &= \max\{\dim \frac{R[It]}{\mathfrak{q}R[t] \cap R[It]} : \mathfrak{q} \in \min(R)\} \\ &= \max\{\dim \left(\frac{R}{\mathfrak{q}} \oplus \frac{I}{\mathfrak{q} \cap I} \oplus \frac{I^2}{\mathfrak{q} \cap I^2} \oplus \ldots\right) : \mathfrak{q} \in \min(R)\} \\ &= \max\{\dim \left(\frac{R}{\mathfrak{q}} \oplus \frac{I + \mathfrak{q}}{\mathfrak{q}} \oplus \frac{I^2 + \mathfrak{q}}{\mathfrak{q}} \oplus \ldots\right) : \mathfrak{q} \in \min(R)\} \\ &= \max\{\dim \frac{R}{\mathfrak{q}} \left[\left(\frac{I + \mathfrak{q}}{\mathfrak{q}}\right) t\right] : \mathfrak{q} \in \min(R)\}.\end{aligned}$$

□

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Recall that we were proving

**Theorem.**

$$\dim R[It] = \begin{cases} \dim R, & I \subseteq \mathfrak{p},\ \forall \mathfrak{p} \in \min(R) \\ \dim R + 1, & \textit{otherwise.} \end{cases}$$

*Proof.* $R$ is a domain. If $I = 0$, then $\dim R[It] = \dim R$, or $I \neq 0$. Let $I = (a_1, \ldots, a_n)$. We'll induct on $n$ to prove that $\dim R[It] \leq \dim R + 1$. If $n = 1$, then $R[It] = R[a_1 t] \cong R[x]$ and $\dim R[x] = \dim R + 1$. Now assume that it is true up through $n - 1$. Let $S = R[a_1 t, \ldots, a_{n-1} t]$. Then $S[x] \overset{\phi}{\twoheadrightarrow} R[It]$ is given by $x \mapsto a_n t$. Then we see that

$$\dim R[It] \overset{\dagger}{<} \dim S[x] = \dim S + 1 \leq \dim R + 1 + 1.$$

To prove †, it is enough to show that $\ker \phi \neq 0$, but $a_i x - a_n(a_i t) \in \ker \phi$ for $i \in \{1, \ldots, n-1\}$. We need $\dim R + 1 \leq \dim R[It]$. Let $\mathfrak{p} := ItR[It] \subseteq R[It]$. Then note that $\frac{R[It]}{\mathfrak{p}} = R \oplus \frac{It}{It} \oplus \frac{I^2 t^2}{I^2 t^2} \oplus \cdots = R$. $\mathfrak{p}$ is prime and $\dim R = \dim \frac{R[It]}{\mathfrak{p}}$. Since $0 \subsetneq \mathfrak{p}$, we see that $\dim R[It] \geq \dim R + 1$. □

**Theorem 19.** *Let $(R, \mathfrak{m})$ be local and suppose that $I \subseteq R$ is an ideal of $R$.*

$$\dim(gr_I(R)) = \dim R.$$

*Proof.*

**Claim 19.1.** *Assume that the theorem holds for a domain.* $\dim(gr_I(R)) = \max\{\dim \frac{R[It]}{\Delta(\mathfrak{q}) + IR[It]} : \mathfrak{q} \in \min(R)\}$. *Let $S = R[It]$. We have*

$$L := \frac{S}{IS} \twoheadrightarrow \frac{S}{\Delta(\mathfrak{q}) + IS} =: D.$$

*Thus,* $\dim L \geq \dim D$.

Let $\mathfrak{q} \in \operatorname{Spec}(S)$ be minimal over $IS$. Then $\mathfrak{q} \supseteq \mathfrak{q}' = \Delta(\mathfrak{q})$, where $\mathfrak{q}' \in \min(S)$ for some $\mathfrak{q} \in \min(R)$. Then we have $\mathfrak{q} \supseteq IS + \Delta(\mathfrak{q})$.

$$\frac{S}{IS + \Delta(\mathfrak{q})} \twoheadrightarrow \frac{S}{\mathfrak{q}}.$$

Thus,

$$\max\left\{\dim \frac{\frac{S}{IS}}{\frac{\mathfrak{q}}{IS}}\right\} \leq \max\left\{\dim \frac{S}{IS+\Delta(\mathfrak{q})}\right\}.$$

$$\begin{aligned}\dim \frac{S}{\Delta(\mathfrak{q})+IS} &= \dim \frac{R[It]}{\mathfrak{q}R[t]\cap S + IS}\\ &= \dim \frac{\frac{R}{\mathfrak{q}} \oplus \frac{I+\mathfrak{q}}{\mathfrak{q}} \oplus \dots}{\frac{I+\mathfrak{q}}{\mathfrak{q}} \oplus \frac{I^2+\mathfrak{q}}{\mathfrak{q}} \oplus \dots}\\ &= \dim(gr_{IR'}R' \text{ for } R' := \frac{R}{\mathfrak{q}}\\ &= \dim R'\\ &= \dim \frac{R}{\mathfrak{q}}.\end{aligned}$$

Now let's prove the theorem for $R$ a domain. Set $S = R[It]$ and define $S_+ := It \oplus I^2t^2 \oplus \dots$ for $I \neq 0$.

**Claim 19.2.** *$S_+ \not\subseteq \sqrt{IS}$.*

Suppose $S_+ \subseteq \sqrt{IS}$. There exists $k$ so that $S_+^k \subseteq IS$, and so $I^kt^k \subseteq I^{k+1}t^k$. $I^k \subseteq I(I^k)$ and by NAK, $I = 0$.

There exists $\mathfrak{q}$ minimal over $IS$ so that $\mathfrak{q} \not\supseteq S_+$. Let $a \in I$ be such that $at \notin \mathfrak{q}$.

**Claim 19.3.** *$\mathfrak{q}$ is a minimal prime over $aS$.*

Assum $aS \subseteq \mathfrak{p} \subsetneq \mathfrak{q}$. Then $a\cdot It \subseteq \mathfrak{p}$ and so $I\cdot at \subseteq \mathfrak{p}$. Thus, $I \subseteq \mathfrak{p} \Longrightarrow IS \subseteq \mathfrak{p}$. I need $\dim \frac{S}{IS} = \dim R$. Then $\dim S = \dim R + 1$ and $\dim \frac{S}{IS} \leq \dim R$. Thus, $\dim \frac{S}{IS} \geq \dim \frac{S}{\mathfrak{q}} = \dim S - \operatorname{ht}\mathfrak{q} = \dim R + 1 - 1 = \dim R$. □

**Lemma 20.** *Suppose that $R \subseteq S$, where $R = R_0[R_1]$ and $S = S_0[S_1]$. If $S$ is a finitely generated $R$-module, then there exists a $k$ so that $(S_+)^{k+1} = R_+(S_+)^k$.*

*Proof.* $S = RS_0 + \dots + RS_k$. Then $S_{k+1} = R_{k+1}S_0 + \dots + R_1S_k \subseteq R_1S_k$. In particular, this is because $R_{k+1}S_0 \subseteq R_1(R_kS_0) \subseteq R_1S_k$. □

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**Exercise 11.** Let $I \subseteq R$ with $\bigcap_{i=0}^{\infty} I^i = 0$. If $\operatorname{gr}_I(R)$ is reduced, then $R$ is reduced. If $\operatorname{gr}_I(R)$ is a domain, then $R$ is a domain. If $\operatorname{gr}_I(R)$ is an integrally closed domain, then $R$ is an integrally closed domain.

normal **Proposition 21.** *Suppose that $I \subseteq R$ and $\bigcap_{i=0}^{\infty} I^i = 0$. If $gr_I(R)$ is reduced, then $\overline{I^n} = I^n$ for all $n > 0$.*

**Corollary 22.** *Let $(R, \mathfrak{m}, k)$ be a local, regular ring. Then $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 0$.*

*Proof.* (of Proposition normal 21)

Assume $\overline{I^n} \supsetneq I^n$ for some $n$ and let $a \in \overline{I^n} \setminus I^n$. Then there exists $k$ so that $a \in I^k \setminus I^{k+1}$ for $k < n$. There exists an equation

$$a^d + r_1a^{d-1} + \dots + r_d = 0$$

with $r_i \in I^{ni}$. Thus, we see that

$$a \in I^n \cdot I^{k(d-1)} + \dots + I^{nd} = \sum_{i=1}^{d} I^{ni+k(d-i)}.$$

Now $0 \neq \overline{a} \in \operatorname{gr}_I(R)$. In particular, $\overline{a} \in \frac{I^k}{I^{k+1}}$. Thus, $0 \neq \overline{a}^d \in \frac{I^{kd}}{I^{kd+1}}$.

It's enough to show that $ni + k(d-i) \geq kd + 1$ for all $i$, which is true if and only if $ni \geq ki + 1$ if and only if $ni > ki$ if and only if $n > k$ (which we know). Thus, $\sum_{i=1}^{d} I^{ni+k(d-i)} \subseteq I^{kd+1}$. □

**Lemma 23.** *Suppose $R = \bigoplus_{n\geq 0} R_n = R[R_1]$ and $S = \bigoplus_{n\geq 0} S_n = S_0[S_1]$ and $R \subseteq S$. If $S_1$ is a finitely generated $R_0$-module, then the containment is module finite if and only if there exists $k > 0$ so that $s_{k+1} = R_{k+1}S_0 + \dots + R_1S_k$.*

**Proposition 24.** $\dim \mathcal{F}_I(R) \leq \dim R$.

**Definition 10.** The *analytic spread of* $I$ is given by $\ell(I) := \dim \mathcal{F}_I(R)$.

*Proof.* Recall that $\mathcal{F}_I(R) = \frac{R[It]}{\mathfrak{m}R[It]}$. Also, $\dim R[It] = d = \dim R$ if $I \subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \min(R)$, or $\dim R[It] = d+1$ if $I \not\subseteq \widetilde{\mathfrak{q}}$ for some $\widetilde{\mathfrak{q}} \in \min(R)$. If $\mathfrak{m}R[It] \subseteq \Delta(\mathfrak{q})$ for all $\mathfrak{q} \in \min(R)$. Note that

$$\begin{aligned} \mathfrak{m} &= \mathfrak{m}R[It] \cap R \\ &\subseteq \mathfrak{q}R[t] \cap R[It] \cap R \\ &\subseteq \mathfrak{q}. \end{aligned}$$

Thus, $I \subseteq \widetilde{\mathfrak{q}}$, which is a contradiction. □

**Proposition 25.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring with* $|k| = \infty$. *If* $I \subseteq R$, *then* $I$ *is integral over an ideal* $J$ *generated by at most* $\ell(I)$ *elements. (*$I \subseteq \overline{J}$*; i.e., there exists* $J \subseteq I$ *so that* $I \subseteq \overline{J}$*.)*

*Proof.* Let $\ell = \ell(I)$. Then $\mathcal{F}_I(R) = \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \ldots$. By the Noether Normalization Lemma, there exist $x_1^*, \ldots, x_\ell^* \in \frac{I}{\mathfrak{m}I}$. Then $k[x_1^*, \ldots, x_\ell^*] \subseteq \mathcal{F}_I(R)$ is module finite. Lift $x_i^*$ to $x_i \in I$ and let $J = (x_1, \ldots, x_\ell) \subseteq I$. Then $\frac{R}{\mathfrak{m}} \oplus \frac{J+\mathfrak{m}I}{\mathfrak{m}I} \oplus \frac{J^2+\mathfrak{m}I^2}{\mathfrak{m}I^2} \cdots \subseteq \mathcal{F}_I(R)$ (which is module finite). There exists $h \in \mathbb{N}$ so that $\frac{I^{h+1}}{\mathfrak{m}I^{h+1}} = \frac{J+\mathfrak{m}I}{\mathfrak{m}I}\left(\frac{I^h}{\mathfrak{m}I^h}\right)$ and by NAK, $I^{h+1} \subseteq JI^h \subseteq I^{h+1}$ and so $JI^h = II^h$. Thus, $I \subseteq \overline{J}$ by the determinant trick. □

**Definition 11.** Suppose that $J \subseteq I$ are ideals. Then $J$ is a *reduction of* $I$ if there exists an $h$ so that $I^{n+1} = JI^n$ for all $n \geq h$.

What we proved shows that there exists $J \subseteq I$ and a reduction such that $\mu(J) \leq \ell(I)$.

reductions **Lemma 26.** *If* $J \subseteq H \subseteq I$ *are ideals, then*

(1) *if* $H \subseteq I$ *is a reduction and* $J \subseteq H$ *is a reduction, then* $J \subseteq I$ *is a reduction.*
(2) $J \subseteq I$ *is a reduction implies that* $H \subseteq I$ *is a reduction*

**Definition 12.** $J \subseteq I$ is a *minimal reduction* if there is no $H \subseteq J$ reduction.

**Definition 13.** If $J \subseteq I$ is a reduction, then the *reduction number* is $r_J(I) := \min\{h : I^{n+1} = JI^n,\ n \geq h\}$.

**Exercise 12.** If $(R, \mathfrak{m})$ is a local and one-dimensional ring, then there exists an $h$ so that for all $J \subseteq I$ reductions (where $I$ is $\mathfrak{m}$-primary), then $I^{n+1} = JI^n$ for all $n \geq h$.

**Exercise 13.** In a 2-dimensional ring, there is a family of ideals $J_n \subseteq I_n$ reductions, but $\lim_{n\to\infty} r_{J_n}(I_n) = \infty$.

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**Example 7.** If $I$ is homogeneous, then $\overline{I}$ is homogeneous. However, if $J \subseteq I$ is a reduction, then $J$ may not be homogeneous.

Let $I = (x^3, xy, y^4) \supseteq J = (xy, x^3 + y^4)$ be ideals of $k[x, y]$. I claim that $J \subseteq I$ is a reduction. It is enough to show that $x^3 \in \overline{J}$. If we do this, then $J \subseteq (J, x^3) = I$ is a reduction. To see that $x^3 \in \overline{J}$, notice that $(x^3)^2 - x^3(x^3 + y^4) + y(xy)^3 = 0$, where $(x^3 + y^4) \in J$ and $(xy)^3 \in J^2$.

**Example 8.** Let $R = \frac{k[x,y]}{xy(x+y)}$ and $I = (x, y)$ and $J = (x + uy)$ for $u$ is a unit of $k$. I claim that $J \subseteq I$ is a reduction. Note that

$$\begin{aligned} JI^2 &= (x + uy)(x^2, xy, y^2) \\ &= (x^3 + uyx^2, x^2y + uxy^2, xy^2 + uy^3) \\ &= (x^3 + u^2xy^2, \underbrace{xy^2 + uxy^2}_{=(1+u)xy^2}, xy^2 + uy^3) \\ &= (x^3, xy^2, y^3) \\ &= (x, y)^3 \\ &= I^3. \end{aligned}$$

**Theorem 27.** *Let $(R, \mathfrak{m}, k)$ be a local ring and let $J \subseteq I$ be a reduction. Then there exists $H \subseteq J$ a reduction that is minimal.*

*Proof.* Let $\Sigma = \{L \subseteq J \text{ reduction}\}$. Note that $\Sigma \neq \emptyset$ because $J \subseteq \Sigma$. Notice also that $\frac{I}{\mathfrak{m}I} \underset{*}{\supseteq} \frac{L+\mathfrak{m}I}{\mathfrak{m}I}$ for $L \in \Sigma$. Let $H \in \Sigma$ be minimal under the inclusion $(*)$. Let $x_1, \dots, x_l \in H$ so that

$$\frac{(x_1, \dots, x_l) + \mathfrak{m}I}{\mathfrak{m}I} = \frac{H + \mathfrak{m}I}{\mathfrak{m}I} \cong \left(\frac{R}{\mathfrak{m}}\right)^l.$$

If $H' = (x_1, \dots, x_l) \subseteq H$, then first I claim that $H'$ is a reduction of $I$. To see this, note that

$$\begin{aligned} I^{n+1} &= HI^n \\ &\subseteq (H + \mathfrak{m}I)I^n \\ &= (H' + \mathfrak{m}I)I^n \\ &= H'I^n + \mathfrak{m}I^{n+1} \\ &\subseteq I^{n+1} + \mathfrak{m}I^{n+1}. \end{aligned}$$

By NAK, we see that $I^{n+1} = H'I^n$.

Now,

$$\left(\tfrac{R}{\mathfrak{m}}\right)^l \quad \cong \quad \tfrac{H'}{\mathfrak{m}H'} \twoheadrightarrow \tfrac{H'+\mathfrak{m}I}{\mathfrak{m}I} \quad \cong \quad \left(\tfrac{R}{\mathfrak{m}}\right)^l.$$

Thus, $H' \cap \mathfrak{m}I = \mathfrak{m}H'$.

I claim now that $H' \subseteq I$ is a minimal reduction. To see this, assume that $H'' \subseteq H'$ is a reduction. Then $H'' + \mathfrak{m}I = H' + \mathfrak{m}I$ and so $H' \subseteq H'' + \mathfrak{m}I$. Now if $a \in H'$ and $b \in H''$ and $y \in \mathfrak{m}I$, with $a = b + y$, then we see that $y = a - b \in H'$. Thus, $H' \subseteq H'' + \mathfrak{m}I \cap H'$ and $H' \subseteq H'' + \mathfrak{m}H'$. By NAK, $H' \subseteq H'' \subseteq H'$ and so equality holds. □

**Corollary 28.** *If $J \subseteq I$ is a minimal reduction, then $J \cap \mathfrak{m}I = \mathfrak{m}J$.*

Recall that Lemma reductions 26 said that if we have $J \subseteq H \subseteq I$ were ideals, then

(1) $J \subseteq I$ is a reduction, then $H \subseteq I$ is a reduction.
(2) $J \subseteq H$ is a reduction and $H \subseteq I$ is a reduction, then $J \subseteq I$ is a reduction.

*Proof.* (of Lemma reductions 26)

(1) $I^{n+1} = JI^n \subseteq HI^n \subseteq I^{n+1}$. Since equality holds on the outside, we get equality everywhere.
(2) $I^{n+1} = HI^n$ for all $n > h$ and $H^{n+1} = JH^n$ for all $n > l$. For all $n > h + l$,

$$\begin{aligned} I^n &= HI^{n-1} \\ &= H^2 I^{n-2} \\ &\;\;\vdots \\ &= H^{n-h} I^h \\ &= J^1 H^{n-h-1} I^h \\ &\;\;\vdots \\ &= J^{n-h-l} H^l I^h \\ &\subseteq J^{n-h-l} I^{l+h} \\ &\subseteq JI^{n-1}. \end{aligned}$$

□

**Proposition 29.** *Suppose that $J \subseteq I$ is a reduction. Then*

(1) $\sqrt{J} = \sqrt{I}$
(2) $\min\left(\frac{R}{J}\right) = \min\left(\frac{R}{J}\right)$

(3) $\mu(J) \geq \ell(I)$.

**Remark 9.** Note that

$$\dim(R) \geq \ell(I) \geq h(J) = h(I).$$

*Proof.* (1) Since $J \subseteq I$, we know that $\sqrt{J} \subseteq \sqrt{I}$ and if $a \in \sqrt{I}$, then $a^n \in I$ and so $(a^n)^l \in I^l = JI^{l-1} \subseteq J$. Thus, $a \in \sqrt{J}$.

(2)

(3) Let $\mathcal{B} = \frac{R}{\mathfrak{m}}\left[\frac{J+\mathfrak{m}I}{\mathfrak{m}I}\right] \subseteq \mathcal{F}_I(R)$. This last inclusion is module-finite. Note that $\dim \mathcal{F}_I(R) = \ell(I)$. Thus, $\dim \mathcal{B} = \ell(I)$. Then $\mathcal{C} = \frac{R}{\mathfrak{m}}\left[x_1, \ldots, x_{\mu(J)}\right] \twoheadrightarrow \mathcal{B}$. Thus, $\dim \mathcal{B} \leq \dim \mathcal{C} = \mu(J)$. □

2/5/07

Recall that for a local ring $(R, \mathfrak{m}, k)$, then

1. if $|k| = \infty$, then $J \subseteq I$ is a minimal reduction then $\mu(J) = \ell(I)$.
2. if $J \subseteq I$ is a minimal reduction, then $J \cap \mathfrak{m}I = \mathfrak{m}J$.
3. a minimal set of generators of $J$ can be extended to a minimal set of generators for $H$ where $J \subseteq H \subseteq I$. (This is true since $\mathfrak{m}J \subseteq J \cap \mathfrak{m}H \subseteq J \cap \mathfrak{m}I = \mathfrak{m}J$ and so $\mathfrak{m}J = J \cap \mathfrak{m}H$.)
4. $\dim R \geq \ell(I) \underset{(|k|=\infty)}{=} \mu(J) \geq \mathrm{ht}(I)$ for some $J \subseteq I$ minimal.

**Exercise 14.** Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring and $M$ a finitely generated $R$-module with $I \subseteq R$ and $\mathfrak{m}$-primary ideal. Then there exists $h > 0$ so that $I^n F_i \cap \Omega_{i+1}(M) \subseteq I^{n-h}\Omega_{i+1}(M)$ for all $i > 0$ and $n > h$, where

$$\ldots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a minimal free resolution.

dimlandht **Theorem 30.** *Let $(R, \mathfrak{m}, k)$ be a local ring with and ideal $I \subseteq R$. Then we have*

$$\dim R \geq \ell(I) \geq ht(I).$$

mingenisl **Theorem 31.** *Suppose $(R, \mathfrak{m}, k)$ is local and $I \subseteq R$. Then there exists $n > 0$ so that $I^n$ has a minimal reduction $J$ minimally generated by $\ell(I)$ elements.*

We won't be giving a proof for Theorem 31.
To prove Theorem 30, use Theorem 31 or the following theorem.

extensionniceness **Theorem 32.** *Let $(R, \mathfrak{m}, k)$ be local. Consider*

$$R \longrightarrow (R[x])_{\mathfrak{m}R[x]} =: S.$$

1. *$R \longrightarrow S$ is a faithfully flat extension and the residue field of $S$ is infinite.*
2. *$J \subseteq I$ is a reduction if and only if $JS \subseteq IS$ is a reduction.*
3. *$\mu(I) = \mu(IS)$*
4. *$ht(I) = ht(IS)$ and therefore, the dimensions are the same.*
5. *$\ell(I) = \ell(IS)$*
6. *$\overline{IR[x]} = \overline{I}R[x]$ and therefore, $\overline{IS} = \overline{I}S$. In particular, $I = \overline{I}$ if and only if $\overline{IS} = \overline{I}S$.*

*Proof.* (of Theorem 32)

(1) Note that

$$R \xrightarrow[\text{flat}]{} R[x] \xrightarrow[\text{flat}]{} R[x]_{\mathfrak{m}R[x]}$$

and $\mathfrak{m}S = \mathfrak{n}$. This means that $R \longrightarrow S$ is faithfully flat.

(2) $\Longrightarrow$: If $J \subseteq I$ is a reduction with $I^{n+1} = JI^n$, then $I^{n+1}S = JSI^nS$.
$\Longleftarrow$: $I^{n+1}S = JSI^nS = JI^nS$. Assume that $JI^n \subsetneq I^{n+1}$. Then we see that $(I^{n+1})/(JI^n) \neq 0$ and so $(I^{n+1})/(JI^n) \otimes_R S \neq 0$, which is a contradiction. Thus, $JI^n \subseteq I^{n+1}$.

(3) Suppose that $I = (a_1, \dots, a_n)$. Then $IS = (a_1, \dots, a_n)S$ and so $\mu(IS) \leq \mu(I)$. Note that there is a map

$$\begin{aligned} R[x]_{\mathfrak{m}R[x]} &\longrightarrow R \\ x &\longmapsto 1 \\ IS &\longmapsto I. \end{aligned}$$

This shows that $\mu(I) \leq \mu(IS)$.

(4) This proof is deferred until next time.

(5) This proof is deferred until next time.

(6) By persistance, $\overline{I}R[x] \subseteq \overline{IR[x]}$. To see "$\supseteq$," note that $\overline{IR[x]}$ is homogeneous with $\deg(x) = 1$ and $\deg(R) = 0$. It's enough to show that $rx^n \in \overline{IR[x]}$, then $r \in \overline{I}$. If $rx^n \in \overline{IR[x]}$, then there exists

$$(rx^n)^m + a_1(rx^n)^{m-1} + \cdots + a_m = 0$$

for $a_i \in I^iR[x]$. Thus,

$$x^{nm}\left(r^m + b_1 r^{m-1} + \cdots + b_m\right) = 0,$$

where $b_i \in I^i$. Thus, $r \in \overline{I}$.

For $\overline{IS} = \overline{I}S$, we show that $W \subseteq R$ a multiplicatively closed subset, we have $\overline{I(W^{-1}R)} = \overline{I}(W^{-1}R)$. To see this, note that persistance tells us that $\overline{I}W^{-1}R \subseteq \overline{IW^{-1}R}$. To see the other direction, note that we have

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

with $a_i \in (IW^{-1}R)^i$. There exists $w_i \in W$ so that $w_i a_i \in I^i$. Let $w = \prod_{i=1}^n w_i$. Then multply the equation by $w^n$ to get

$$(wr)^n + w^n a_1 r^{n-1} + \cdots + w^n a_n = 0$$

Then we see that there exists $u_1 \in W$ so that $u_1$ times the left-hand side is 0 in $W^{-1}R$. By multiplying by $u_1^n$, you show that $u_1 wr \in \overline{I}$ and so $r \in \overline{I}W^{-1}R$.

□

**Example 9.** Suppose that $J \subseteq I$ is a reduction. There exists an integer $r_J(I)$ so that $I^{n+1} = JI^n$ for all $n > r_J(I)$. Set $r(I) = \min\{r_J(I) : J \subseteq I \text{ is a reduction}\}$. A question that many people have been working on is what is $r(I)$.

Let $I \subseteq R$ be a homogeneous ideal where $R = k[x_1, \dots, x_n]$. Let $m_>(I)$ be the initial ideal of $I$ in a given term order $>$. Then $r(I) \leq r(m_>(I))$. Unfortunately, there are examples where strict inequality holds and so the number is not determined by the Hilbert function.

The good news is that if $(R, \mathfrak{m}, k)$ is a local Cohen-Macaulay ring of depth $d$ and $I \subseteq R$ is $\mathfrak{m}$-primary and $J \subseteq I$ is a reduction generated by a regular sequence. If

$$\operatorname{depth}(\operatorname{gr}_I(R)) \geq d - 1$$

then

$$r_J(I) \leq e_1 - e_0 + \ell\left({}^{(R)}/_{(I)}\right) + 1,$$

where $e_1$ and $e_0$ come from the Hilbert polynomial of $I$.

**Questions:** What are the depths of $\operatorname{gr}_I(R)$ and of $R[It]$? To answer thiss we are going to first discuss superficial elements an superficial sequences and then Rees's theorem about multiplicity and finally we'll discuss Rees's characterization of analytically unramified rings.

**Exercise 15.** Read in Atiyah-MacDonald all you can about the Hilbert function.

2/7/07

The setup from last time was that we had the following

$$\begin{array}{ccccc} R & \longrightarrow & R[x] & \longrightarrow & R[x]_{\mathfrak{m}R[x]} =: S \\ k & & & & k(x) \end{array}$$

Let $Q \in \operatorname{Spec}(S)$ and $Q' \in \operatorname{Spec}(R[x])$ be such that $Q'S = Q$. Let $\mathfrak{q} = Q' \cap R$. Note that $\text{ht}Q' = \text{ht}Q$. To see that $\text{ht}\mathfrak{q} = \text{ht}Q$, we will show that $\text{ht}\mathfrak{q} = \text{ht}Q$. Consider

$$R_{\mathfrak{q}} \longrightarrow R_{\mathfrak{q}}[x] \longrightarrow R_{\mathfrak{q}}[x]_{\mathfrak{m}[x]}.$$

Then we see that

$$\begin{aligned} \text{ht}Q' &= \dim R_{\mathfrak{q}}[x]_{Q'} \\ &\leq \dim R_{\mathfrak{q}}[x] \\ &= \dim R_{\mathfrak{q}} + 1. \end{aligned}$$

If $\text{ht}Q' = \dim R_{\mathfrak{q}} + 1 = d + 1$ (where $d := \dim R_{\mathfrak{q}}$), then

$$Q_0 \subsetneq \cdots \subsetneq Q_{d+1} = Q' \subsetneq (\mathfrak{m}R[x], x) = (\mathfrak{m}, x)$$

and so $\dim R_{\mathfrak{q}}[x] = d + 2$, which is a contradiction. If $\text{ht}Q' = \text{ht}\mathfrak{q} = d$, then

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{q} \subseteq R.$$

Then we see that

$$\mathfrak{q}_0[x] \subsetneq \cdots \subsetneq \mathfrak{q}_d[x] = \mathfrak{q}[x] \subseteq Q'.$$

Since $Q'$ and $\mathfrak{q}$ have the same height, we see that $Q' = \mathfrak{q}[x]$.

It follows that $\text{ht}(I) = \text{ht}(IS)$ for all $I \subseteq R$. We also need to show that $\ell(I) = \ell(IS)$.

$$\begin{aligned} \ell(I) &= \dim \mathcal{F}_I(R) \\ &= \dim \left( \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \dots \right). \end{aligned}$$

We see that

$$\begin{aligned} \mathcal{F}_I(R) \otimes_R S &= \frac{S}{\mathfrak{m}S} \oplus \frac{IS}{\mathfrak{m}IS} \oplus \dots \\ &= \mathcal{F}_{IS}(S). \end{aligned}$$

Notice that $\mathcal{F}_{IS}(S)$ is flat over $\mathcal{F}_I(R)$. Thus, we see that $\dim \mathcal{F}_{IS}(S) = \dim \mathcal{F}_I(R) + 0$. In general, $(R, \mathfrak{m}) \underset{\text{flat}}{\longrightarrow} (S, \mathfrak{n})$, then $\dim(S) = \dim(R) + \dim \frac{S}{\mathfrak{m}S}$.

**Definition 14.** Let $I \subseteq R$ and $M$ be an $R$-module. An element $x \in I$ is *superficial* (in $I$) with respect to $M$ if there exists $c \geq 0$ so that for every $n > c$ the following holds:

$$\left(I^{n+1}M :_M x\right) \cap I^c M = I^n M.$$

Note that it is not clear that superficial elements exist, but we will show that they do exist if the residue field is infinite.

**Remark 10.** If $x \in I$ is superficial (in $I$) for $M$, then $x^m \in I^m$ is superficial (in $I^m$) for $M$.

**Lemma 33.** *If $x \in I$ and $x$ is a non-zero divisor on $M$, then $x$ is superficial with respect to $M$ if and only if there exists $c \in \mathbb{N}$ so that $\left(I^{n+1}M :_M x\right) = I^n M$ for all $n \geq c$.*

*Proof.* $\Longleftarrow$: This direction is easy.

$\Longrightarrow$: Recall that the Artin-Rees Lemma says that if $N \subseteq M$ and $I \subseteq R$, then there exists $k$ so that $I^n M \cap N = I^{n-k}(I^k M \cap N) \subseteq I^{n-k} N$ for all $n \geq k$. By the Artin-Rees Lemma applied to $xM \subseteq M$, we have

$$\begin{aligned} xI^{n-k} &\supseteq I^n M \cap xM \\ &= x\left(I^n M :_M x\right) \end{aligned}$$

Since $x$ is a non-zero divisor on $M$, we see that $(I^n M :_M x) \subseteq I^{n-k} M$. Let $n > k + c$, where $c$ satisfies $\left(I^{n+1}M : x\right) \cap I^c M = I^n M$. Thus,

$$\begin{aligned} I^n M = \left(I^{n+1}M : x\right) \cap I^c M &\subseteq I^{n-k+1} M \cap I^c M \\ &= I^{n-k+1} M. \end{aligned}$$

Somehow we need to get $k$ to go away. ∎

**Lemma 34.** *Assume that $I$ contains a non-zero divisor on $M$. Then every superficial element in $I$ with respect to $M$ is a non-zero divisor on $M$.*

*Proof.* Let $x$ be superficial. Then there exists $c$ so that

$$\left(I^{n+1}M :_M x\right) \cap I^c M = I^n M.$$

Note that $(0 :_M x)I^c \subseteq (I^{n+1}M : x) \cap I^c M = I^n M$ for all $n \geq c$. Then $(0 :_M x)I^c \subseteq \bigcap_n I^n M = 0$. Thus, $(0 :_M x)I^c = 0$. Let $y \in I$ be a non-zero divisor on $M$. Then we see that $y^c$ is a non-zero divisor on $M$ and so $y^c(0 :_M x) = 0$ means that $(0 :_M x) = 0$. □

2/9/07

To fix the Theorem from last time, we had $(I^n M : x) \subseteq I^{n-k}$ and for $n > k + c$ we have $I^{n-k}M \subseteq I^c M$. In particular,

$$(I^n M : x) \subseteq (I^n M : x) \cap I^c M = I^{n-1}M.$$

**Theorem 35.** *Let $(R, \mathfrak{m}, k)$ be local with $|k| = \infty$ and $I \subseteq R$ be an ideal of $R$. Suppose that $M$ is a finitely generated $R$-module. Then there exists $x \in I$ a superficial element with respect to $M$.*

This last line means:

> There exists an open set in $\frac{I}{\mathfrak{m}I}$ from which you can choose $\overline{x}$ which lifts to $x$ that is superficial in $I$ with respect to $M$.

*Proof.* Consider $\mathcal{M} = \operatorname{gr}_I(M)$. Let $0 = N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_m$ be a primary decomposition of zero. Set $P_i = \operatorname{Ass}\left((\mathcal{M})/(N_i)\right)$ for $1 \leq i \leq s$ and for $i \geq s+1$, set $P_i = \sqrt{\left(N_i :_{\operatorname{gr}_I(R)} \mathcal{M}\right)}$. Then we have $\frac{I}{I^2} \subseteq P_i$ for each $1 \leq i \leq s$. However, for $i \geq s+1$, we have $\frac{I}{I^2} \not\subseteq P_i$. There exists $c \geq 0$ so that $\frac{I^c M}{I^{c+1}M} \subsetneq N_i$ for all $i = 1, \ldots, s$. (Take $c$ to be the maximum of the $n_i$ so that $P_i^{n_i} \subseteq \left(N_i :_{\operatorname{gr}_I(R)} \mathcal{M}\right)$ for $i = 1, \ldots, s$.) Pick $\overline{x} \in \frac{I}{\mathfrak{m}I} \setminus \left(\bigcup_{i=s+1}^n \frac{P_i + \mathfrak{m}I}{\mathfrak{m}I}\right)$.

I claim that $(I^n M : x) \cap I^c M = I^{n-1}M$ for $n - 1 \geq c$. Assume not; i.e., assume that $(I^n M : x) \cap I^c M \supsetneq I^{n-1}M$ for $n - 1 \geq c$ and take $y \in (I^n M : x) \setminus I^{n-1}M$. There exists $k$ so that $y \in I^k M$ but $y \notin I^{k+1}M$. Then $c \leq k \leq n-1$ and

$$\frac{x + I^2}{I^2} \cdot \frac{y + I^{k+1}M}{I^{k+1}M} = \frac{xy + I^{k+2}M}{I^{k+2}M} = 0$$

Now $xy \in I^n M$ for $n \geq k+2$ and $\frac{xy + I^{k+2}M}{I^{k+2}M} \in N_{s+1} \cap \cdots \cap N_m$, which means that $\overline{y} \in N_{s+1} \cap \cdots \cap N_m$. Now since $y \in I^k M \subseteq I^c M$ and so $\overline{y} \in \frac{I^c M}{I^{c+1}M} \subseteq N_i$ for $i = 1, \ldots, s$. Thus, $\overline{y} \in N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_n = 0$ and so $y \in I^{k+1}M$, which contradicts the fact that $y \notin I^{k+1}M$. □

**Definition 15.** The sequence $x_1, \ldots, x_s \in I$ is a *superficial sequence in $I$ with respect to $M$* if $x_1$ is superficial with respect to $M$ and for each $i \in \{2, \ldots, s\}$, then $\overline{x_i}$ in $\frac{I}{(}x_1, \ldots, x_{i-1})$ is superficial with respect to $\frac{M}{(x_1, \ldots, x_{i-1})M}$.

uperficialseq

**Lemma 36.** *Let $(x_1, \ldots, x_s) \in I$ be a superficial sequence with respect to $M$. Then $I^n M \cap (x_1, \ldots, x_s M = (x_1, \ldots, x_s)I^{n-1}M$ for all $n \gg 0$.*

*Proof.* We will induct on $s$. If $s = 1$, then $I^n M \cap x_1 M = x_1 I^{n-1}M$ for $n \gg 0$. By the Artin-Rees Lemma, there exists $k$ so tath $I^n M \cap x_1 M \subseteq x_1 I^{n-k}M \subseteq I^c M$ (for $n > k + c$). Thus, when $n > k + c$ we have

$$I^n M \cap x_1 M = x_1(I^n M :_M x_1) \cap x_1 I^c M.$$

Let $y \in I^n M \cap x_1 M$. Write $y = x_1 a$ for $a \in (I^n M : x_1)$ and $y = x_1 b$ where $b \in I^{n-k}M \subseteq I^c M$. Then $x_1(a - b) = 0$ and so $a - b \in (0 :_M x_1) \subseteq (I^n M : x_1)$. In particular, $b = a - (a - b) \in (I^n M : x_1) \cap I^c M$. Thus, $y \in x_1\left[(I^n M : x_1) \cap I^c M\right] = x_1 I^{n-1}M$.

Now assume that we know the case $s - 1$. Then $I^n M + (x_1, \ldots, x_{s-1})M) \cap (x_1, \ldots, x_s)M = x_s I^{n-1}M + (x_1, \ldots, x_{s-1})M$, but then

$$\begin{aligned} I^n M \cap (x_1, \ldots, x_s)M &\subseteq x_s I^{n-1}M + (x_1, \ldots, x_{s-1})M \cap I^n M \\ &= x_s I^{n-1}M + (x_1, \ldots, x_{s-1})I^{n-1}M \\ &= (x_1, \ldots, x_s)I^{n-1}M. \end{aligned}$$

□

2/12/07

**Corollary 37.** *Let $(R, \mathfrak{m}, k)$ be local and $I \subseteq R$ and $M$ a finitely generated $R$-module. Suppose that $x \in I$ is superficial with respect to $M$. Then*

$$\left[\frac{gr_I(M)}{x^* gr_I(M)}\right]_n = [gr_{I'}(M')]_n$$

*for $n \gg 0$ and $I' = \frac{I}{x}$ and $M' = \frac{M}{xM}$ as $\frac{R}{x}$-modules and $x^* = \frac{x+I^{n+1}}{I^{n+1}} \in gr_I(R)$ if $x \in I^n \setminus I^{n+1}$.*

Note that $x \in \mathrm{gr}_I(R)$ implies that there exists an $n$ so that $x^* \in (I^n)/(I^{n+1})$ where $x^* = x + I^{n+1}$ is such that $x \in I^n \setminus I^{n+1}$.

*Proof.*

$$\begin{aligned}\left[\frac{\mathrm{gr}_I(M)}{x^*\mathrm{gr}_I(M)}\right]_n &= \left(\frac{\dfrac{I^nM}{I^{n+1}M}}{\dfrac{xI^{n-1}M + I^{n+1}M}{I^{n+1}M}}\right)\\ &= \left(\frac{I^nM}{xI^{n-1}M + I^{n+1}M}\right).\end{aligned}$$

$$\begin{aligned}[\mathrm{gr}_{I'}(M')]_n &= \frac{(I')^nM'}{(I')^{n+1}M'}\\ &= \frac{(I^nM + xM)/(xM)}{(I^{n+1}M + xM)/(xM)}\\ &\cong \frac{I^nM + xM}{I^{n+1}M + xM}\\ &\cong \frac{I^nM}{(I^{n+1}M + xM) \cap I^nM}\\ &\underset{I^{n+1}M \subseteq I^nM}{=} \frac{I^nM}{(I^{n+1}M \cap I^nM) + (xM \cap I^nM)}\\ &= \frac{I^nM}{I^{n+1}M + (xM \cap I^nM)}.\end{aligned}$$

As $xI^{n-1}M + I^{n+1}M \subseteq I^{n+1}M + (xM \cap I^nM)$, there exists $\phi$ a surjection

$$\begin{array}{ccc}\phi : \frac{I^nM}{xI^{n-1}+I^{n+1}M} & \twoheadrightarrow & \frac{I^nM}{I^{n+1}M+(xM\cap I^nM)}\\ \| & & \|\\ (\mathrm{gr}_I(M))_n & & (\mathrm{gr}_{I'}(M'))_n\,,\end{array}$$

where $\ker\phi = \frac{I^{n+1}M+(xM\cap I^nM)}{I^{n+1}M+xI^{n-1}M} = 0$. As $x$ is superficial in $I$ with respect to $M$ by Lemma superficialseq 36, we see that $xM \cap I^nM = xI^{n-1}M$ for $n \gg 0$. Thus, $\ker\phi = \frac{I^{n+1}M+xI^{n-1}M}{I^{n+1}M+xI^{n-1}M} = 0$. Thus, $\phi$ is an isomorphism and so

$$[\mathrm{gr}_{I'}(M')]_n = \left[\frac{\mathrm{gr}_I(M)}{x^*\mathrm{gr}_I(M)}\right]_n.$$

□

**Theorem 38.** *Let $(R, \mathfrak{m}, k)$ be local and $|k| = \infty$. Suppose $I \subseteq R$ is an ideal and $\ell = \ell(I)$. Then there exists $(x_1, \ldots, x_\ell) = J \subset I$ is a minimal reduction such that $x_1, \ldots, x_\ell$ is a superficial sequence (with respect to $R$).*

*Proof.* We will proceed by induction on $\ell$. If $\ell = 0$, then $\mathcal{F}_I(R) = \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2}$. This says that $I$ is nilpotent. If $\ell > 0$, and $P_1, \ldots, P_n$ are the minimal primes in $\mathcal{F}_I(R)$, then $\frac{I}{\mathfrak{m}I} \not\subseteq \bigcup_i P_i$. Then there exists $\overline{x} \in \frac{I}{\mathfrak{m}I} \setminus \bigcup_i P_i$. Lift $\overline{x}$ to $x \in I$ and consider $J = \frac{I}{x}$. Then we see that

$$\frac{\mathcal{F}_I}{x\mathcal{F}_I} \twoheadrightarrow \mathcal{F}_J(R)$$

$$\frac{(I)/(\mathfrak{m}I)}{(xI+\mathfrak{m}I)/(\mathfrak{m}I)} = \frac{I}{\mathfrak{m}I+xI} \twoheadrightarrow \frac{\frac{I}{x}}{\mathfrak{m}\frac{I}{x}} = \frac{I}{\mathfrak{m}I+x}$$

Now $\dim \mathcal{F}_J(\frac{R}{x}) \leq \dim (\mathcal{F}_I)/(x\mathcal{F}_I) = \dim \mathcal{F}_I - 1$. If $s = \ell(J)$, then we have $s < \ell$. There exist $x'_1, \ldots, x'_s \in \frac{R}{x}$ with $x'_i \in J$ so that $(x'_1, \ldots, x'_s)$ is a superficial sequence and a reduction for $J$. Thus, $J^{n+1} = (x'_1, \ldots, x'_s)J^n$ for all GET NOTES. □

Our goal now is to characterize analytically unramified rings.

**Definition 16.** A local ring $(R, \mathfrak{m}, k)$ is *analytically unramified* if $\hat{R}$ is reduced; i.e., $\bigcap_{\mathfrak{p}\in\min(\hat{R})} \mathfrak{p} = 0$.

Notice that if $\hat{R}$ is reduced, then since $R \hookrightarrow \hat{R}$ means that $R$ has to be reduced. Tehere are examples of $R$ reduced, but $\hat{R}$ not reduced.

**Theorem 39.** *Let $R$ be a complete local domain (Noetherian). Let*

$$\begin{array}{ccc} R & \hookrightarrow & \mathbb{Q}(R) \\ & \searrow & \\ & S & \Big\downarrow \text{ finite} \\ & & \searrow \\ & & L, \end{array}$$

*where $S$ is the integral closure of $R$ in $L$. Then $S$ is a finitely generated $R$-module and is complete.*

We won't prove this last theorem.

anunram **Theorem 40** (Rees)**.** *Let $(R, \mathfrak{m}, k)$ be local and Noetherian. The following are equivalent:*

1. *$R$ is analytically unramified*
2. *for every $\mathfrak{m}$-primary ideal $I$, there exists $h > 0$ so that $\overline{I^{n+h}} \subseteq I^n$ for all $n > 0$*
3. *there exists an $\mathfrak{m}$-primary ideal $J$ so that there exists $h > 0$ with $\overline{J^{n+h}} \subset J^n$*
4. *there exists $f : \mathbb{N} \longrightarrow \mathbb{N}$ so that $f(n) \underset{n\to\infty}{\longrightarrow} \infty$ and $J$ an $\mathfrak{m}$-primary ideal so that $\overline{J^{n+f(n)}} \subseteq J^n$ for all $n > 0$.*

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We will show how to reduce the previous theorem to the following theorem.

noname **Theorem 41.** *Let $(R, \mathfrak{m}, k)$ be a local complete domain. Then the integral closure of $R$ in $\mathbb{Q}(R)$ is a finitely generated $R$-module.*

**Theorem 42** (Lipman-Sathaye)**.** *If $R$ is regular of dimension $d$, then $\overline{I^{n+d}} \subseteq I^{n+1}$ for all $n > 0$ and for all $I \subseteq R$.*

**Theorem** (Huneke)**.** *GET NOTES.*

**Example 10.** Let $R = \frac{k[x,y,z]}{z^2-x^5-y^7}$ and note that $\dim R = 2$. Let $I = (x, y)^2$. If $z \in \overline{I}$, then $z^2 - x^5 - y^7 = 0$ and this is in $(x, y)^4$. Furthermore, $z \in \overline{(x, y)^{2=0+d}}$ GET NOTES.

**Lemma 43.** *Let $(R, \mathfrak{m}, k)$ be local and $I$ be $\mathfrak{m}$-primary. Then $\overline{I\hat{R}} = \overline{I}\hat{R}$.*

*Proof.* The easy case is done by persistence: $\overline{I}\hat{R} \subseteq \overline{I\hat{R}}$. For the other direction, let $x \in \overline{I\hat{R}}$ and note that there exists an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where $a_i \in I^i\hat{R}$. Furthermore, we know that there exists $s$ so that $\mathfrak{m}^s \subseteq I$.

There exists $y \in R$ and $b_i \in I^i$ so that $x - y \in \mathfrak{m}^{ns}\hat{R}$ and $a_i - b_i \in \mathfrak{m}^{ns}\hat{R}$. This means that $x = y + b$ and $a_i = b_i + b_i'$. Plugging this into the equation for integral closure, we have

$$y^n + b_1y^{n-1} + \cdots + b_n \in \mathfrak{m}^{ns}\hat{R} \cap R = \mathfrak{m}^{ns} \subseteq I^n.$$

Let $c_n = y^n + b_1y^{n-1} + \cdots + b_n \in I^n$ and note that

$$y^n + b_1y^{n-1} + \cdots + (b_n - c_n) = 0 \Longrightarrow y \in \overline{I},$$

since $b_n - c_n \in I^n$. Thus, $x = x - y + y \in \mathfrak{m}^{ns}\hat{R} + \overline{I}\hat{R} \subseteq I^n\hat{R} + \overline{I}\hat{R} = \overline{I}\hat{R}$. ☐

*Proof.* (of Theorem anunram 40)

Note that $2 \Longrightarrow 3 \Longrightarrow 4$. Now assuming 4, let's see that 1 holds. Let $\{P_1, \ldots, P_n\} \subseteq \min(\hat{R})$. Let $N = \bigcap_{i=1}^n P_i$ and note that $N \subseteq \overline{I}$ for all $I \subseteq R$. In particular,

$$N \subseteq \bigcap_{n\geq 0} \overline{J^{n+f(n)}\hat{R}} = \bigcap_{n\geq 0} \overline{J^{n+f(n)}}\hat{R} \subseteq \bigcap_{n\geq 0} J^n\hat{R} = 0.$$

To see $1 \Longrightarrow 2$, since $N = 0$ reduce to the completion. $\overline{I^{n+k}} \subseteq I^n$. It's enough to show that $\overline{I^{n+k}}\hat{R} \subseteq I^n\hat{R}$. Note that $\overline{I^{n+k}}\hat{R} = \overline{I^{n+k}\hat{R}} = \overline{\left(I\hat{R}\right)^{n+k}}$. Reduce now to a domain. Let $\{P_1, \ldots, P_n\} = \min(\hat{R})$. For all $i = 1, \ldots, n$, write $(-)_i$ for $\pmod{P}_i$. Assume that for each $i = 1, \ldots, n$ there exists $k_i$ so that

$$\overline{I_i^{n+k_i}} \subseteq I_i^n$$

for all $n > 0$. Let $k = \max\{k_1, \ldots, k_n\}$. In $R$,

$$\overline{I^{n+k}} \subseteq I^n + P_i$$

for all $i = 1, \ldots, n$. Let $c_i \in P_1 \cap \cdots \cap \widehat{P_i} \cap \cdots \cap P_n \setminus P_i$ and set $c = c_1 + \cdots + c_n$. Note that $c \in \bigcup_{i=1}^n P_i$.

Then $c_i\overline{I^{n+k}} = (c_1 + \cdots + c_n)\overline{I^{n+k}} \subseteq \sum\left(c_i\overline{I^{n+k}}\right) \subseteq \sum(c_iI^n) \subseteq I^n$. Also, $c_i\overline{I^{n+k}} \subseteq c_iI^n + c_iP_i = c_iI^n$ since $c_iP_i \in P_1 \cap \cdots \cap P_n = 0$. Thus, $c\overline{I^{n+k}} \subseteq I^n$. There exists $h$ so that $I^n \cap (c) \subseteq cI^{n-h}$. Then

$$\not{c}\overline{I^{n+k+h}} \subseteq I^{n+h} \subseteq \not{c}I^n \Longrightarrow \overline{I^{n+k+h}} \subseteq I^n.$$

It is enough to show that if $R$ is a complete local domain and $I \subseteq R$ is $\mathfrak{m}$-primary, then there exists $k$ so that $\overline{I^n} = I^{n-k}\overline{I^k}$. Let $S = R[\![a_1t, \ldots, a_st]\!] \subseteq R[\![t]\!]$ for $I = (a_1, \ldots, a_s)$. If $M = \{\sum a_it^i : a_0 \in \mathfrak{m}\}$, then I have the following claims:

**Claim 43.1.** *$M$ is maximal and $S$ is complete with the $M$-topology.*

**Claim 43.2.** *If $T$ is the integral closure of $S$ in $R[\![t]\!]$, then $T = \prod_{n\geq 0} \overline{I^n}t^n$.*

Assuming these claims, then by the theorem, $T$ is module finite over $S$ there exists $k$ so that $\overline{I^{n+k}} = I^n\overline{I^k}$. ☐

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**Proposition 44.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Then*

1. *if $R$ is analytically unramified, then $(R)/(\mathfrak{p})$ is analytically unramified for all $\mathfrak{p} \in \min(R)$*
2. *if $(R)/(\mathfrak{p})$ is analytically unramified, then for all $\mathfrak{p} \in \min(R)$ and $R$ is reduced, then $R$ is analytically unramified.*

*Proof.* (1) GET NOTES

(2) We need $\hat{R}$ reduced, and so we need $0 = \bigcap_{i=1}^{l} Q_i$ for $Q_i \in \operatorname{Spec}(\hat{R})$. As $R$ is reduced, we see that

$$0 = \bigcap_{P \in \min(R)} P$$
$$\Longrightarrow 0 = \bigcap_{P \in \min(R)} P\hat{R},$$

but for all $P \in \min(R)$, we see that

$$\frac{\hat{R}}{P\hat{R}}$$

is reduced. Thus, $P\hat{R} = \bigcap_{i=1}^{n_p} Q_{P_i}$, where $Q_{P_i} \in \operatorname{Spec}(\hat{R})$ minimal over $P\hat{R}$, which implies that $0 = \bigcap_{P_i \in \min(R)} Q_{P_i}$. □

polyunram **Corollary 45.** *If $R$ is analytically unramified, then the integral closure of $R[It]$ in $R[t]$ is finitely generated as an $R[It]$-module.*

unramalg **Theorem 46.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain with $|k| = \infty$. The following are equivalent:*

(1) *$R$ is analytically unramified.*
(2) *For each finitely generated $R$-algebra $S$ such that $R \subset S \subset \mathbb{Q}(R)$, the integral closure of $S$ in $\mathbb{Q}(R)$ is finite over $S$.*

*Proof.* (of Corollary polyunram 45)
There exists $h$ so that $\overline{I^{n+h}} \subseteq I^n$ for all $n > 0$ and so

$$\overline{R[It]} = \underbrace{R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots}_{S_{n+h}} \oplus \overline{I^{n+h}}t^{n+h}.$$

Note that $\overline{I^{n+h}}t^{n+h} \subseteq I^n t^n$. Thus, we see that $\overline{R[It]} \subseteq R[It]S_{n+h}$. Now use Exercise (unramequal 16) to finish the proof. □

unramequal **Exercise 16.** Let $R$ be analytically unramified. For $I \subseteq R$, there exists $h$ so that $\overline{I^{n+h}} = I^n \overline{I^h}$ for all $n \geq 0$.

modfine **Theorem 47.** *If $R$ is analytically unramified, then $\overline{R} \subseteq \mathbb{Q}(R)$ is module finite over $R$.*

Note that Theorem noname 41 $\Longrightarrow$ Theorem modfine 47 $\Longrightarrow$ Theorem unramalg 46.

*Proof.* (of Theorem unramalg 46)
$1 \Longrightarrow 2$: By Theorem modfine 47, $\overline{R} \subseteq \mathbb{Q}(R)$ is module finite over $R$. Assume $\overline{R}$ is generated by $\frac{x_1}{r}, \dots, \frac{x_n}{r}$. Thus, $r\overline{R} \subseteq R$. Let $S$ be an $R$-algebra with $R \subset S \subset \mathbb{Q}(R)$ with $S = R[\frac{y_1}{y}, \dots, \frac{y_d}{y}]$. Set $I = (y_1, \dots, y_d)$. Then note that $S = \bigcup_{s \geq 0} \frac{I^s}{y^s}$.

**Claim 47.1.** *I claim that $\overline{S} \subseteq S \cdot \frac{1}{ry^h}$. Let $z \in \overline{S}$. Then there exists*

zintegral (2) $$z^n + s_1 z^{n-1} + \dots + s_n = 0,$$

*for $s_i \in S$ and $s_i = \frac{a_i}{y^l}$ and $a_i \in I^l$ for $l > h$.*

Multiply Equation (zintegral 2) by $y^{ln}$. Then we get

ztimesint (3) $$(y^l z)^n + a_1 (zy^l)^{n-1} + \dots + a_n (y^l)^{n-1} = 0,$$

for $a_i (y^l)^{i-1} \in I^l \cdot (I^l)^{i-1} \in I^{li} \subset R$. Thus, $y^l z \in \overline{R}$ and $ry^l z \in R$. Now multiply Equation (ztimesint 3) by $r^n$ and we see that $ry^l z \in \overline{I^l}$. Thus, $ry^l z \in \overline{I^l \overline{R}} \cap R = \overline{I^l} \subseteq I^{l-h}$. Now $R \longrightarrow \overline{R}$ is module finite. Thus, $ry^h z \in \frac{I^{l-h}}{y^{l-h}} \subset S$, and so $z \in S \cdot \frac{1}{ry^k}$. Now $\overline{I^{n+h}} \subseteq I^n$ for all $n \geq 0$ and $\overline{I^n} \subseteq I^{n-h}$ for all $n \geq h$. □

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Let us now complete the proof of Theorem unramalg 46.

*Proof.* $2 \implies 1$: It is enough to show that there exists an $\mathfrak{m}$-primary ideal $J$ and $h > 0$ so that $\overline{J^{n+h}} \subseteq J^n$ for $n > 0$. Let $J = \mathfrak{m}$ and note that there exists $x \in \mathfrak{m}$ superficial for $R$, and so

$$(\mathfrak{m}^n : x) \cap \mathfrak{m}^c = \mathfrak{m}^{n-1} \qquad n \gg 0 \iff (\mathfrak{m}^n : x) = \mathfrak{m}^{n-1} \qquad n \gg 0.$$

Let $S = R\left[\frac{\mathfrak{m}}{x}\right] = R\left[\frac{x_1}{x}, \dots, \frac{x_l}{x}\right] \subset \mathbb{Q}(R)$, when $\mathfrak{m} = (x_1, \dots, x_l)$. Then $\overline{S}$ is a finitely generated $S$-module and $\overline{S} \cap R_x$ is a finitely generated $S$-module. There exists $p > 0$ so that $\overline{S} \cap R_x \subseteq S \cdot \frac{1}{x^p}$. Let $u \in \overline{\mathfrak{m}^n}$. Then there exists an equation

uclose (4)
$$u^q + a_1 u^{q-1} + \cdots + a_q = 0,$$

for $a_i \in (\mathfrak{m}^n)^i$. Multiplying equation (4) by $\frac{1}{x^{qn}}$, we get

$$\left(\frac{u}{x^n}\right)^q + \left(\frac{a_1}{x^n}\right)\left(\frac{u}{x^n}\right)^{n-1} + \cdots + \frac{a_q}{x^{nq}} = 0,$$

where $\frac{a_i}{x^{ni}} \in \frac{\mathfrak{m}^n}{x^{ni}}$. Then note that $\frac{u}{x^n} \in \overline{S} \cap R_x \subseteq S \cdot \frac{1}{x^p}$. Now we can write

$$\frac{u}{x^n} = \frac{v}{x^{m+p}},$$

where $v \in \mathfrak{m}^m$. Thus, $ux^{m+p} = x^n v \in \mathfrak{m}^{n+m}$ and so $u \in (\mathfrak{m}^{n+m} : x^{m+p}) \subseteq \mathfrak{m}^{n+m-m-p} = \mathfrak{m}^{n-p}$. This is where

$$(\mathfrak{m}^{n+m} : x) = \mathfrak{m}^{n+m-1}$$
$$(\mathfrak{m}^{n+m} : x^2) = \mathfrak{m}^{n+m-1-1}$$

and $ax^2 \implies (ax) \in (\mathfrak{m}^{n+m} : x) = \mathfrak{m}^{n+m-1} \implies a \in (\mathfrak{m}^{n+m-1} : x) = \mathfrak{m}^{n+m-1-1}$. □

reducedfinite **Theorem 48.** *Let $(R, \mathfrak{m}, k)$ be a local domain. If $\hat{R}$ is reduced, then $\overline{R}$ is finite over $R$.*

reducedcompletefinite **Proposition 49.** *If $R$ is reduced and complete, then $\overline{R}$ is module finite over $R$.*

*Proof.* (of Proposition 49)

Let $\{P_1, \dots, P_s\} = \min(R)$. Then $R \subset \underbrace{\frac{R}{P_1} \times \cdots \times \frac{R}{P_s}}_{\mathcal{R}} \subseteq \mathbb{Q}\left(\frac{R}{P_1}\right) \times \cdots \times \mathbb{Q}\left(\frac{R}{P_s}\right) = \mathbb{Q}(R)$. $\mathcal{R}$ is module finite over $R$ and so $\mathcal{R}$ is integral over $R$.

firstclaim **Claim 49.1.** *I claim that $\overline{R} = \overline{\mathcal{R}}$.*

secclaim **Claim 49.2.** *I claim that $\overline{R} = \overline{\frac{R}{P_1}} \times \cdots \times \overline{\frac{R}{P_s}}$.*

Assume Claim 49.2 is true. Then each $\overline{\frac{R}{P_i}}$ is finitely generated over $\frac{R}{P_i}$. Thus, $\overline{\frac{R}{P_1}} \times \cdots \times \overline{\frac{R}{P_s}}$ is finitely generated over $\frac{R}{P_1} \times \cdots \times \frac{R}{P_s}$. Thus, $\overline{\mathcal{R}} = \overline{R}$ is finitely generated over $\frac{R}{P_1} \times \cdots \times \frac{R}{P_s}$ and hence is fintiely generated over $R$. □

*Proof.* (of Theorem 48) Note that $R \subset \overline{R} \subseteq \mathbb{Q}(R)$. Thus, the containment $\hat{R} \subseteq \overline{\hat{R}}$ is module finite and $\overline{\hat{R}} \subseteq \mathbb{Q}(\hat{R})$. Thus,

$$\overline{R} \otimes_R \hat{R} \subseteq \overline{\hat{R}} \implies \overline{R} \otimes_R \hat{R}$$

is module finite over $\hat{R}$. So $\overline{R} \otimes_R \hat{R} \otimes_{\hat{R}} {}^{(\hat{R})}/_{(\mathfrak{m}\hat{R})} = k^l$ since $\overline{R} \otimes_R \hat{R} = \overline{R} \otimes_R {}^{(R)}/_{(\mathfrak{m})} = k^l$. □

rlrcontain **Theorem 50.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local regular ring of characteristic $p > 0$ and suppose that $I \subseteq R$ is generated by $n$ elements. Then $\overline{I^{m+n}} \subseteq I^{m+1}$ for all $m \geq 0$.*

**Definition 17.** Let $R$ be Noetherian of characteristic $p > 0$ with $I \subseteq R$. The *tight closure of $I$* is

$$I^* = \left\{ x \in R : \exists c \in R \setminus \left( \bigcup_{P \in \min(R)} P \right) \text{ so that } cx^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0 \right\}.$$

Recall that $I^{[p^e]} = \{u^{p^e} : u \in I\}$. Then if $I = (u_1, \dots, u_n)$, then $I^{[p^e]} = (u_1^{p^e}, \dots, u_n^{p^e})$ and $(u+v)^p = u^p + v^p$.

rlrstar **Theorem 51.** *Let $(R, \mathfrak{m})$ be a local regular ring. Then $I = I^*$ for all $I \subseteq R$.*

**Remark 11.** Let $x \in \overline{I}$. Then there exists an equation

$$x^l + a_1 x^{l-1} + \cdots + a_l = 0$$

for $x^l \in I(x+I)^{l-1} = (I+x)^l$. By induction, $(I+x)^n = I^{n-l+1}(I+x)^{l-1}$, and in particular, $x^n \in I^{n-l+1}$.
Notice that $x^l \cdot x^n = x^{l+n} \subseteq I^{l+n-l+1} = I^{n+1} \subseteq I^n$. Let $c = x^l$ and then $c \cdot x^n \in I^n$ for all $n \geq 0$.

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**Theorem 52.** *Let $(R, \mathfrak{m})$ be local of characteristic $p > 0$. Suppose $I = (u_1, \ldots, u_n)$ and $ht(I) > 0$. Then $\overline{I^{n+m}} \subseteq \left(I^{m+1}\right)^*$ for all $m > 0$.*

As a corollary, Theorem 51 (rlrstar) implies Theorem 50 (rlrcontain).

*Proof.* Set $J = I^{n+m}$. Assume that $\overline{J} \not\subseteq \left(I^{m+1}\right)^* \cup P_1 \cup \cdots \cup P_s$, where $\{P_i : 1 \leq i \leq s\} = \min(R)$. There exists $x \in \overline{J} \setminus (P_1 \cup \cdots \cup P_s)$. By a remark, we know that there exists $l > 0$ so that $\underbrace{x^l}_{c} x^k \in J^k$ for all $k > 0$. Then $cx^k \in J^k$ for all $k > 0$ and $J^k = \left(I^{n+m}\right)^k$. Note that $J^k$ is generated by $u_1^{b_1}, \ldots, u_n^{b_n}$, where $\sum_{i=1}^n b_i = nk + mk$.

**Claim 52.1.** $I^{nk+mk} \subseteq \left(u_1^k, \ldots, u_n^k\right)^{m+1}$.

*Proof.* Set $a_i = \lfloor \frac{b_i}{k} \rfloor$ and note that $a_i + 1 > \frac{b_i}{k}$. Then

$$\sum_{i=1}^n (a_i + 1) > \sum_{i=1}^n \frac{b_i}{k} = m + n.$$

Thus, $\sum_{i=1}^n a_i \geq m+1$. Let $\overline{u} = \prod_{i=1}^n u_i^{ka_i} \in \left(u_1^k, \ldots, u_n^k\right)^{m+1}$. On the other hand, $ka_i \leq b_i$ means that $\overline{u} \Big| u_1^{b_1} \ldots u_n^{b_n}$ and so GET NOTES.

Set $k = p^l = q$ for all $q > 0$. Then $cx^q \in J^q \subseteq (u_1^q, \ldots, u_n^q)^{m+1} = \left[(u_1, \ldots, u_n)^{m+1}\right]^{[q]}$. Thus, we see that $x \in \left((u_1, \ldots, u_n)^{m+1}\right)^*$. ☐

☐

**Exercise 17.** Let $R \xrightarrow{\phi} S$ be flat and $x \in R$ and $I \subseteq R$ an ideal. Then $(I :_R x) S = (IS :_S \phi(x))$.

**Theorem** (Kunz)**.** *If $R$ is a regular ring of characteristic $p > 0$, then $F : R \longrightarrow R$ given by $x \mapsto x^p$ is flat.*

*Proof.* (of Theorem 51 (rlrstar))
Fix $I \subseteq R$. Note that $R \longrightarrow R$ given by $x \mapsto x^q$ (where $q = p^e$) is flat. By the exercise,

$$\left(I^{[q]} : x^q\right) = (I : x)^{[q]}.$$

Let $x \in I^*$. There exists $c \neq 0$ so that $cx^q \in I^{[q]}$ for all $q \gg 0$. Then $c \in (I^{[q]} : x^q) = (I : x)^{[q]}$ for all $q \gg 0$. If $x \notin I$, then $(I : x) \subseteq \mathfrak{m}$. Thus, $(I : x)^{[q]} \subseteq \mathfrak{m}^{[q]} \subseteq \mathfrak{m}^q$. Thus, $c \in \bigcap_q \mathfrak{m}^q = 0$, which is a contradiction. ☐

If $R$ is analytically unramified, then for each $I$ that is $\mathfrak{m}$-primary, there exists an $h$ so that $\overline{I^{n+h}} \subseteq I^n$. The question is whether or not we can choose $h$ to be independent of $I$. The answer is that indeed you can! To do this, you must assume that the characteristic of the ring is $p > 0$ or that $\mathbb{Q} \subseteq R$. If $R$ isn't regular, then $h \neq d - 1$.

## 4. Multiplicity and Integral Closure

4.1. **Hilbert Polynomials.** Let $(R,\mathfrak{m})$ be local. Suppose that $I$ is $\mathfrak{m}$-primary and $M$ is finitely generated over $R$. Then $\frac{M}{I^nM}$ has finite length. To see this, let $h$ be such that $\mathfrak{m}^h \subseteq I$. Then we see that $\mathfrak{m}^{nh}\left(\frac{M}{I^nM}\right) = 0$ and so $\frac{M}{I^nM}$ has finite length.
**Question:** What is $\lambda\left(\frac{M}{I^nM}\right)$?

hilbert **Theorem 53.** *Let $(R,\mathfrak{m})$ be local and suppose that $I$ is $\mathfrak{m}$-primary and $M$ is finitely generated over $R$. There exists a polynomial $f(t) \in \mathbb{Q}[t]$ so that*

$$f(n) = \lambda\left(\frac{M}{I^nM}\right)$$

*for all $n \gg 0$ and $\deg(f) = \dim(M) \leq \dim(R)$.*

*Proof.* We will induct on $\dim(M)$. If $\dim(M) = 0$, then $\dim \frac{R}{\operatorname{ann}(M)} = 0$. Thus, there exists an $h$ so that $\mathfrak{m}^h \subseteq \operatorname{ann}(M)$. Then there exists $s$ so that $I^s \subseteq \operatorname{ann}(M)$. Thus, $\lambda\left(\frac{M}{I^nM}\right) = \lambda(M)$ for all $n > s$ and $f(t) = \lambda(M)$.

Now assume that $\dim(M) > 0$. Assume that the theorem holds for $\dim(M) - 1$ and let $x$ be superficial in $I$ with respect to $M$. Then $x \notin \bigcup_{P \in \min(R)} P$. Note that we need to reduce to the case where $|k| = \infty$. Consider the map $R \hookrightarrow R(x) = R[x]_{\mathfrak{m}[x]}$ and then $\lambda\left(\frac{M}{IM}\right) = \lambda\left(\frac{M \otimes_R R(x)}{IR(x)M \otimes_R R(x)}\right)$. Thus, we may assume that $|k| = \infty$.

Next time we will do the induction step. □

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Recall that we were proving Theorem hilbert 53 and needed to finish the inductive step.

*Proof.* Pick $x \in I$ that is superficial for $M$ with $x \notin \bigcup_{P \in \min(M)} P$. Then consider the exact sequence coming from the (almost) multiplication map by $x$:

$$0 \longrightarrow \frac{I^mM}{I^{n-1}M} \longrightarrow \frac{M}{I^{n-1}M} \xrightarrow{x} \frac{M}{I^nM} \longrightarrow \frac{M}{xM + I^nM} \longrightarrow 0.$$

Then we see that

$$\underbrace{\lambda\left(\frac{M}{I^nM}\right) - \lambda\left(\frac{M}{I^{n-1}M}\right)}_{f(n)} = \lambda\left(\frac{M}{xM + I^nM}\right) - \lambda\left(\frac{(I^nM : x)}{I^{n-1}M}\right).$$

**Claim 53.1.** *$f(n)$ is degree $d-1$ for $n \gg 0$.*

**Exercise 18.** Show that $g(n) = \lambda\left(\frac{M}{I^nM}\right)$ is a polynomial of degree $d$ for $n \gg 0$.

colonslarge **Claim 53.2.** *$(I^nM : x) = (0 :_M x) + I^{n-1}M$ and $(0 :_M x) \cap I^{n-1}M = 0$.*

Assume the claim. Then we see that $\frac{(I^nM:x)}{I^{n-1}M} \cong (0 :_M x)$.

*Proof.* (of Claim colonslarge 53.2) By the Artin Rees Lemma,

$$x(I^nM : x) \subseteq xM \cap I^nM \subseteq xI^{n-k}M.$$

For every $u \in (I^nM : x)$, there exists $v \in I^{n-k}M$ so that $xu = xv$ and so $x(u-v) = 0$ and hence $u - v \in (0 :_M x)$. Since $u = (u-v) + v$, we see that $u \in (0 :_M x) + I^{n-k}M$. Then if $n \geq k + c$, we see that

$$(I^nM : x) \subseteq [(0 : x) + I^cM]$$

and

$$\begin{aligned}(I^nM : x) &\subseteq ((0 : x) + I^cM) \cap (I^nM : x)\\ &\subseteq (0 :_M x) + I^cM \cap (I^nM : x)\\ &= (0 :_M x) + I^{n-1}M.\end{aligned}$$

□

Now for $n > c$, note that

$$(0:_M x) \cap I^n M \subseteq (0:_M x) \cap I^c M$$

and

$$\left(\bigcap_l (I^l M : x)\right) \cap I^c M \subseteq \bigcap_l \left[(I^l M : x) \cap I^c M\right] \subseteq \bigcap_l I^{l-1} M = 0.$$

□

**Definition 18.** Let $d = \dim(R)$ and let $P_{I,M}$ be the Hilbert-Samuel polynomial. Then the *multiplicity* is given by $e(I, M) = d! a_d$, where $a_d$ is the coefficient of the degree $d$ part of $P_{I,M}$.

Notice that if $\dim(M) < d$, then $e(I, M) = 0$. Also, we should note that this isn't a completely standard definition. If $I = \mathfrak{m}$, then we write $e(M)$ to mean $e(\mathfrak{m}, M)$. If $M$ is $R$, then we write $e_I$ and we write $e(R)$ for $e(\mathfrak{m}, R)$, and finally if we write $e(J)$ this is $e(\mathfrak{m}, J)$.

**Remark 12.** Let $f : \mathbb{N} \longrightarrow \mathbb{Q}$ be a polynomial $f(t) \in \mathbb{Q}[t]$ such that $f(n) \in \mathbb{N}$ for all $n \gg 0$, then $f$ can be written $f(n) = \sum_{i=0}^{d} a_i \binom{n+i}{i}$ with $a_i \in \mathbb{N}$.

**Lemma 54.** *Let $(R, \mathfrak{m})$ be $d$-dimensional and $\dim(M) = d$. Let $I \subseteq R$ be $\mathfrak{m}$-primary. Let $x \in R$ and denote by $R'$ the ring $\frac{R}{xR}$ and set $M' = \frac{M}{xM}$ and finally $I' = IR'$. If $\dim(R') = d - 1$, then $e_{R'}(I', M') \geq e_R(I, M)$.*

*Proof.* Note that $\dim(M') = d - 1$. Then consider

$$0 \longrightarrow \frac{(I^n M : x)}{I^{n-1} M} \longrightarrow \frac{M}{I^{n-1} M} \xrightarrow{x} \frac{M}{I^n M} \longrightarrow \frac{M}{xM + I^n M} \longrightarrow 0.$$

$$\frac{M}{xM + I^n M} = \frac{M'}{I^n M'}$$

For all $n \gg 0$, we have

$$\begin{aligned} g(n) = \lambda\left(\frac{M'}{I^n M'}\right) &= \lambda\left(\frac{M}{I^n M}\right) - \lambda\left(\frac{M}{I^{n-1} M}\right) + \lambda\left(\frac{(I^n M : x)}{I^{n-1} M}\right) \\ &= P_{I,M}(n) - P_{I,M}(n-1) + f(n) \\ &= \frac{e_R(I, M)}{d!} n^d + a_{d-1} n^{d-1} + \cdots - \frac{e_R(I, M)}{d!}(n-1)^d - a_{d-1}(n-1)^d + \cdots + f(n). \end{aligned}$$

GET NOTES. □

lengthbinom **Proposition 55.** *Let $\dim(R) = d$. Let $I = (x_1, \ldots, x_d) \subseteq R$ and $I$ be $\mathfrak{m}$-primary. Let $M$ be a finitely generated $R$-module. Then*

(1) $\lambda\left(\frac{M}{I^n M}\right) \leq \lambda\left(\frac{M}{IM}\right) \cdot \binom{n+d-1}{d}$
(2) *If $R$ is Cohen Macaulay, then*

$$\lambda\left(\frac{R}{I^n R}\right) = \lambda\left(\frac{R}{IR}\right) \binom{n+d-1}{d}$$

*and $e_I(R) = \lambda\left(\frac{R}{IR}\right)$.*

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*Proof.* (of Proposition lengthbinom 55)

$$\begin{aligned} \lambda\left(\frac{M}{I^n M}\right) &= \sum_{i=1}^{n-1} \lambda\left(\frac{I^i M}{I^{i+1} M}\right) \\ &\leq \sum \mu(I^i) \lambda\left(\frac{M}{IM}\right) \\ &\leq \lambda\left(\frac{M}{IM}\right) \sum_{i=0}^{n-1} \binom{i+d-1}{d}, \end{aligned}$$

where the first inequality holds because

$$\begin{array}{ccc} R^{\mu(I^i)} & \twoheadrightarrow & I^i \\ \left(\frac{M}{IM}\right)^{\mu(I^i)} & \twoheadrightarrow & I^i \otimes \frac{M}{IM} \\ & & \downarrow \\ & & \frac{I^i M}{I^{i+1}M}. \end{array}$$

To see the second part of the proposition, then $x_1, \ldots, x_d$ is a regular sequence and therefore, $\frac{I^i}{I^{i+1}}$ is a free module over $\frac{R}{I}$ of rank $\binom{i+d-1}{d}$ and

$$\frac{R}{I} \oplus \frac{I}{I^2} \oplus \ldots \cong \frac{R}{I}[X_1, \ldots, X_d].$$

□

**Lemma 56.** *Let*

$$0 \longrightarrow N \longrightarrow H \longrightarrow M \longrightarrow 0$$

*be a short exact sequence. Suppose $I \subseteq R$. Then $e(I, N) + e(I, M) = e(I, H)$.*

*Proof.* Tensor the short exact sequence with $\frac{R}{I^n}$ and get

$$\frac{N}{I^n N} \longrightarrow \frac{H}{I^n H} \longrightarrow \frac{M}{I^n M} \longrightarrow 0.$$

The Artin-Rees lemma says that there exists $h$ so that $I^n H \cap N \subseteq I^{n-h} N$ for all $n \geq h$. Thus,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{N}{I^n H \cap N} & \longrightarrow & \frac{H}{I^n H} & \longrightarrow & \frac{M}{I^n M} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \frac{N}{I^{n-h} N}. & & & & & & \end{array}$$

Thus, we get

$$\lambda\left(\frac{N}{I^{n-h}N}\right) + \lambda\left(\frac{M}{I^n M}\right) \leq \lambda\left(\frac{H}{I^n H}\right) \leq \lambda\left(\frac{N}{I^n N}\right) + \lambda\left(\frac{M}{I^n M}\right),$$

which gives us

$$\frac{e_R(I,N)}{d!}(n-h)^d + \cdots + \frac{e_R(I,M)n^d}{d!} + \cdots \leq \frac{e_R(I,H)n^d}{d!} + \cdots \leq \left(\frac{e_R(I,N)}{d!} + \frac{e_R(I,M)}{d!}\right)n^d + \ldots,$$

and comparing coefficients of $n^d$, we get the equality we desired. □

**Proposition 57.** *Let $I \subseteq R$ be $\mathfrak{m}$-primary. Let $\Lambda = \{P \in \operatorname{Spec}(R) : \dim \frac{R}{P} = \dim R\} \subseteq \min(R)$. Let $M$ be a finitely generated $R$-module. Then*

$$e_R(I, M) = \sum_{P \in \Lambda} e_R(I, \frac{R}{P})\lambda(M_P).$$

*(Note that $e_R(I, \frac{R}{P}) = e_{\frac{R}{P}}(I, \frac{R}{P})$.)*

*Proof.* Let $0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_0 = M$ so that $\frac{M_i}{M_{i+1}} \cong \frac{R}{P}$ for some $P \in \operatorname{Spec}(R)$. Note that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{R}{P_1} & \hookrightarrow & M & \longrightarrow & M^2 & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & M_1 & & & & \frac{M}{M_1} & & \\
 & & & & & & & & \\
0 & \longrightarrow & \frac{R}{P_2} & \longrightarrow & M' & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \frac{R}{P_m} & \xrightarrow{\cong} & M_{n-1} & \longrightarrow & 0 & &
\end{array}$$

Thus, we see that $e(I, M) = \sum_{P \in L} e(I, \frac{R}{P})$, where $L = \{P_0, \ldots, P_{n-1}\}$ (has repetition). We also see

$$e(I, M) = \sum_{P \in L \cap \Lambda} e(I, \frac{R}{P}),$$

where $P \in \Lambda$ appears in the set $L$ $\lambda(M_P)$ times. Hence,

$$e(I, M) = \sum_{P \in \Lambda} e(I, \frac{R}{P}) \lambda(M_P).$$

$\square$

**Proposition 58.** *Let $(R, \mathfrak{m})$ be local, with $I$ and $J$ $\mathfrak{m}$-primary. If $\overline{J} = \overline{I}$, then $e(I) = e(J)$.*

*Proof.* It is enough to show the proposition with $I = \overline{J}$ (note $J \subseteq \overline{J}$ and $I \subseteq \overline{I}$). Then $e(J) = e(\overline{J}) = e(\overline{I}) = e(I)$. Thus, it is enough to show that $e(J) = e(\overline{J})$. For then $J \subseteq \overline{J} = I$ and there exists $n$ so that $I^n = J^{n-h} I^h \subseteq J^{n-h}$. Thus, we get that $\frac{R}{I^n} \twoheadrightarrow \frac{R}{J^n}$ and so $J \subseteq I$ and $J^n \subseteq I^n$ and $I^n = J^{n-h} I^h \subseteq J^{n-h}$, which means

$$\lambda\left(\frac{R}{J^{n-h}}\right) \leq \lambda\left(\frac{R}{I^n}\right) \leq \lambda\left(\frac{R}{J^n}\right),$$

and

$$\frac{e_R(J, R)}{d!}(n-h)^d + \cdots \leq \frac{e_R(I, R)}{d!} n^d + \cdots \leq \frac{e_R(J, R)}{d!} n^d + \ldots,$$

which means that $e_R(I, R) = e_R(J, R)$. $\square$

equidim

**Theorem 59.** *Let $(R, \mathfrak{m})$ be local such that $\hat{R}$ is equidimensional. Suppose $I \subseteq J$ are $\mathfrak{m}$-primary ideals. If $e(I) = e(J)$, then $J \subseteq \overline{I}$ (and hence $\overline{I} = \overline{J}$).*

*Proof.* First reduce to the case where $|k| = \infty$. Next, reduct to a complete ring. Finally, reduce to a domain. (This is all to get the inductive step to work.)

We will prove this by induction on $\dim(R)$. If $\dim(R) = 0$, then every ideal has the same integral closure. This is because $\mathfrak{m}^n = 0$ and so $\overline{I} = \mathfrak{m}$.

Now suppose $\dim(R) > 0$. We may assume $R = \hat{R}$. Since $I\hat{R} \subseteq J\hat{R}$ and $e_{\hat{R}}(I\hat{R}) = e(I)$ (since $\lambda\left(\frac{R}{I^n}\right) = \lambda\left(\frac{\hat{R}}{I^n \hat{R}}\right)$ for all $I \subseteq \hat{R}$ and all $n$). If the theorem is true in the completion, then $J\hat{R} = \overline{I\hat{R}}$ and so $J = J\hat{R} \cap R \subseteq \overline{I\hat{R}} \cap R = \overline{I}\hat{R} \cap R = \overline{I}$. $\square$

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Recall that we were proving Theorem equidim 59.

*Proof.* By the associativity formula with $\min(R) = \{P_1, \ldots, P_n\}$we have

$$\begin{aligned} e(I) &= e_R(I, R) \\ &= \sum_{i=1}^{n} e_R(I, \frac{R}{P_i})\lambda(R_{P_i}) \\ &= \sum_{i=1}^{n} e_{\frac{R}{P_i}}(\frac{IR}{P_i}, \frac{R}{P_i})\lambda(R_{P_i}) \\ &= \sum_{i=1}^{n} e_R(J, \frac{R}{P_i})\lambda(R_{P_i}) \end{aligned}$$

GET NOTES.

for all $i = 1, \ldots, n$. Thus, $e(\frac{IR}{P_i}) = e(\frac{JR}{P_i})$ if $\frac{JR}{P_i} \subseteq \overline{\frac{IR}{P_i}}$ for each $i = 1, \ldots, n$. Thus, $J \subseteq \overline{I}$ due to Proposition first 1. Since $R$ is a complete local domain, we see that $R$ is analytically unramified. Then $\overline{R[It]}$ is $R[It]$-finite and $\overline{R[Jt]}$ is $R[Jt]$-finite. In particular, there exist $h$ and $l$ so that $\overline{I^{nl}} = \overline{I^l}^n$ and $\overline{I^{hn}} = \overline{I^h}^n$. Substitute $\overline{I^{hl}}$ for $I$ and $\overline{J^{hl}}$ for $J$.

GET NOTES.

Then we see that $e(\overline{I^{hl}}) = e(I^{hl}) = e(J^{he}) = e(\overline{J^{hl}})$. Assume that $\overline{J^{hl}} = \overline{I^{hl}}$. Then we see that $J \subseteq \overline{I}$. For $j \in J$ we see that $j^{hl} \in \overline{I^{hl}}$ and so there exists an equation

$$j^{hln} + a_1(j^{hl})^{n-1} + \cdots + a_n = 0$$

for $a_i \in I^{hli}$. This means that $j \in \overline{I}$. Because $|k| = \infty$, let $x \in I \setminus I^2$ be superficial in $I$ for $R$. Thus, we have $(I^n : x) = I^{n-1}$ for $n > c$. By replacing $I$ with $I^c$ and $x$ with $x^c$ we get $(I^{cn} : x^c) = (I^c)^{n-1}$. Denote by $R' = \frac{R}{x}$. Then we see that $I' := IR' \subseteq JR' =: J'$ and $e(I') \geq e(J') \geq e(J) = e(I)$ But $\frac{(I^n:x)}{I^{n-1}} = 0$ and so $e(I') = e(I)$. Thus, $e(I') = e(J')$. By induction on $\dim(R)$ we get $J' \subseteq \overline{I'}$.

**Claim 59.1.** *$x^* \in gr_J(R) = \frac{R}{J} \oplus \frac{J}{J^2} \oplus \ldots$ is a non-zerodivisor in $gr_J(R)$ of degree one (that is, $x \in J \setminus J^2$).*

If we assume the claim, then we get $J' \subseteq \overline{I'}$ and so $\overline{I'} = \overline{J'}$ thus $I' \subseteq J'$ is a reduction and so there exists $n$ such that $(J')^n = I'(J')^{n-1}$. Thus, we see that $J^n \subseteq IJ^{n-1} + xR$ and so $J^n \subseteq IJ + xR \cap J^n = IJ^{n-1} + x(J^n : x)$. Thus, we see that $J^n \subseteq IJ^{n-1} + xJ^{n-1} \subseteq IJ^{n-1}$ because the claim implies that $(J^n : x) = J^{n-1}$. Therefore, $I \subseteq J$ is a reduction and so $\overline{I} = \overline{J}$.

The reason why the claim implies that $(J^n : x) = J^{n-1}$ because the kernel of the map $\frac{J^{n-2}}{J^{n-1}} \xrightarrow{\cdot x} \frac{J^{n-1}}{J^n}$ is exactly $(J^n : x)$ but as $x$ is a non-zerodivisor in $\mathrm{gr}_J(R)$ we see that $(J^n : x) = J^{n-1}$.

*Proof.* (of Claim)

First, suppose that $x^*$ doesn't have degree one in $\mathrm{gr}_J(R)$. Then we see that $x \in J^2$. Then we see that

$$\left(\frac{(J^n : x)}{J^{n-1}}\right) \supseteq \left(\frac{J^{n-2}}{J^{n-1}}\right) \Longrightarrow \lambda\left(\frac{(J^n : x)}{J^{n-1}}\right) \geq \lambda\left(\frac{J^{n-2}}{J^{n-1}}\right).$$

But we know that $\frac{J^{n-2}}{J^{n-1}}$ grows as a polynomial of degree $d-1$. This is a contradiction since $e(J, \frac{R}{x}) = e(J') = e(J)$.

To see that $x^*$ is a non-zerodivisor, consider $(0 :_{\mathrm{gr}_J(R)} x^*)$. Then if we look at the degree $n$ piece, we have

$$\left[(0 :_{\mathrm{gr}_J(R)} x)\right]_n = \frac{(J^{n+1} : x)}{J^n} \subseteq \frac{(J^{n+1} : x)}{J^n}.$$

Since $e(J') = e(J)$, we see that $\lambda\left(\frac{(J^{n+1}:x)}{J^n}.\right)$ grows with a polynomial of degree at most $d - 2$. Thus, $\dim_{\mathrm{gr}_J(R)}(0 : x^*) \leq d - 1$. Note that $\dim_{\mathrm{gr}_J(R)}\left(\frac{\mathrm{gr}_J(R)}{(0:(0:x^*))}\right) \leq d - 1$. By Lemma gradedass 60, we see that

$$(0 : (0 : x^*)) \nsubseteq \bigcup_{Q \in \mathrm{Ass}(\mathrm{gr}_J(R))} Q \Longrightarrow (0 : x^*) = 0.$$

□

□

The steps in the above proof were as follows

(1) $R$ is complete.
(2) $|k|$ is infinite.
(3) Reduce to the case when $R$ is a domain.
(4) $I, J$ are normal.
(5) $(I^n : x) = I^{n-1}$ for all $n > 0$.

gradedass **Lemma 60.**
$$Ass(gr_J(R)) = \min(gr_J(R))$$
*and for every* $Q \in \min(gr_J(R))$ *we have* $\dim \frac{gr_J(R)}{Q} = d$.

3/2/07 Recall that we were about to start proving Lemma gradedass 60.

*Proof.* Note that $\mathrm{gr}_J(R) = \frac{R[Jt,t^{-1}]}{t^{-1}R[Jt,t^{-1}]}$. Suppose that $Q' \in \mathrm{Ass}(\mathrm{gr}_J(R))$ and lift this to $Q$ in $S = R[Jt, t^{-1}]$. It is enough to show that $Q \in \min\left(\frac{S}{t^{-1}S}\right)$, or that $\mathrm{ht}(Q) = 1$. Then $S = R[Jt, t^{-1}] \subseteq T = R[t, t^{-1}]$. Also, $S = \overline{S} \subset T$. If $Q \in \mathrm{Ass}\left(\frac{S}{t^{-1}S}\right)$ and there exists $y \in S$ so that $Q = \left(t^{-1}S :_S y\right)$. Localize at $Q$ and set $S = S_Q$ and $T = T_Q$ and note that $S \subseteq T$ with $Q$ maximal. Then $\frac{y}{t^{-1}}Q \subseteq S$. Assume that $\frac{y}{t^{-1}}Q \subseteq Q$. This implies that $\frac{y}{t^{-1}} \in \overline{S} = S$, which is a contradiction. Thus, $\frac{y}{t^{-1}}Q = S$, which means that there exists $z$ so that $\frac{y}{t^{-1}}z = 1 \Longrightarrow zy = t^{-1}$. Therefore, $Q = (yzS :_S y) = zS$ and so $\mathrm{ht}(Q) = 1$.

Now $R$ is a complete domain and so $R$ is "universally catenary," which means that for each $Q \in \mathrm{Spec}(S)$ we have $\mathrm{ht}(Q) + \dim \frac{S}{Q} = \dim(S) + 1 = d + 1$. (?? GET NOTES... IS THIS FORMULA RIGHT??) □

Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Consider the Koszul complex

$$K(\mathbf{x}; R): \quad 0 \to R^{\binom{n}{n}} \to R^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{2}} \to R^{\binom{n}{1}} \to R \to 0$$

$$R^{\binom{n}{i}} \longrightarrow R^{\binom{n}{i-1}}$$
$$e_{k_1 \ldots k_i} \longmapsto \sum_{h=1}^{i} (-1)^{h+1} x_{k_n} e_{k_1 \ldots \hat{k_h} \ldots k_i}$$

Denote by $H_i(\mathbf{x}; R) = H_i(K(\mathbf{x}; R))$. Then we have

(1) $H_0(\mathbf{x}; R) = \frac{R}{\mathbf{x}}$
(2) $I = (\mathbf{x}) = (\mathbf{x}, y)$ then $H_1(\mathbf{x}, y; R) = H_1(\mathbf{x}; R) \oplus H_0(\mathbf{x}; R)$.
(3) $I = (\mathbf{x}) \subseteq \mathrm{ann} H_i(\mathbf{x}; R) \subseteq \sqrt{I}$.
(4) there exists a long exact sequence
$$\xrightarrow{\cdot y} H_i(\mathbf{x}; R) \longrightarrow H_i(\mathbf{x}, y; R) \longrightarrow H_{i-1}(\mathbf{x}; R) \xrightarrow{\cdot y} H_{i-1}(\mathbf{x}; R) \longrightarrow \cdots$$
(5) Let $h = \max\{i : H_i(\mathbf{x}; R) \neq 0\}$ implies that $\mathrm{grade}(I) = n - h$, where $I = (\mathbf{x}) = (x_1, \ldots, x_n)$.

**Exercise 19.** Find an ideal $I = (x_1, \ldots, x_n)$ and an $i$ so that $\mathrm{ann}(H_i(\mathbf{x}; R)) \not\subseteq \mathrm{ann} H_{i+1}(\mathbf{x}; R)$.

It is known that there exists $R$, $M$ and $I = (\mathbf{x})$ so that $\mathrm{ann}(H_i(\mathbf{x}; M)) \not\subseteq \mathrm{ann}(H_{i+1}(\mathbf{x}; M))$.

**Question:** Do $I = (\mathbf{x})$ and $\mathrm{ann}(H_1(\mathbf{x}; R))$ have the same integral closure? In general, the answer is no. What happens in the case of $\mathfrak{m}$-primary ideals?

**Theorem 61.** *Let* $(R, \mathfrak{m})$ *be local. Suppose* $I$ *is an ideal with* $I = \overline{I}$ *and is* $\mathfrak{m}$*-primary such that* $0 \neq H_1(\mathbf{x}; R)$. *Then* $ann(H_1(\mathbf{x}; R)) = I$.

**Lemma 62.** *Let* $J = (\mathbf{y}) \subseteq R$ *be an ideal. Let* $c, x \in R$ *be such that* $(J, cx)$ *is* $\mathfrak{m}$*-primary. Then* $\lambda\left(H_i(J, c)\right) = \lambda\left(anna_c H_i(J, cx)\right)$, *where* $H_i(J, c) = H_i(\mathbf{y}, c; R)$ *and* $H_i(J, cx) = H_i(\mathbf{y}, cx; R)$ *and* $ann_c\left(H_i(J, cx)\right) = \left(0 :_{H_i(\mathbf{y}, cx; R)} c\right)$.

*Proof.* Note that $(J, cx) \subseteq (J, c)$. We will proceed by induction on $i$. If $i = 0$, then

$$0 \longrightarrow \frac{((J,cx):_R c)}{(J,cx)} \longrightarrow \frac{R}{(J,cx)} \xrightarrow{\cdot c} \frac{R}{(J,cx)} \longrightarrow \frac{R}{(J,c)} \longrightarrow 0.$$

Thus, we see that

$$\begin{aligned} \lambda\left(H_0((J,c);R)\right) &= \lambda\left(\frac{R}{(J,c)}\right) \\ &= \lambda\left(\frac{((J,cx):c)}{(J,cx)}\right) \\ &= \lambda\left(\text{ann}_c H_0(J,cx;R)\right). \end{aligned}$$

Looking at the long exact sequence and cutting out a short exact sequence from it, we have

$$\cdots \longrightarrow H_i(J,cx;R) \xrightarrow{\cdot c} H_i(J,cx;R) \longrightarrow H_i(J,cx,c;R) \longrightarrow H_{i-1}(J,cx;R) \xrightarrow{\cdot c} \cdots$$

$$0 \longrightarrow \frac{H_i(J,cx;R)}{cH_i(J,cx;R)} \longrightarrow H_i(J,cx,c;R) \longrightarrow \text{ann}_c H_{i-1}(J,cx;R) \longrightarrow 0$$

Thus, we have

$$\begin{aligned} \lambda\left(H_i(J,c)\right) + \lambda\left(H_{i-1}(J,c)\right) &= \lambda\left(H_i(J,cx,c;R)\right) \\ &= \lambda\left(\text{ann}_c H_{i-1}(J,cx;R)\right) + \lambda\left(\frac{H_i(J,cx;R)}{cH_i(J,cx;R)}\right). \end{aligned}$$

This means that

$$\lambda\left(H_i(J,c;R)\right) = \lambda\left(\frac{H_i(J,cx;R)}{cH_i(J,cx;R)}\right).$$

If $M$ has finite length, then $\lambda\left(\frac{M}{cM}\right) = \lambda\left(\text{ann}_c M\right)$ and

$$0 \longrightarrow \text{ann}_c M \longrightarrow M \xrightarrow{\cdot c} M \longrightarrow \frac{M}{cM} \longrightarrow 0.$$

$\square$

We'll prove that if $I$ is $\mathfrak{m}$-primary, then $\mu(I) \geq d+1$. Let $c \in \text{ann} H_1(I;R) \setminus I$, then $(I :_R c) = (\mathfrak{m}I :_R c)$ and so $I = (x_1, \ldots, x_n)$ is minimally generated and so $(x_1, \ldots, x_n, c)$ is the generating set for $(\mathbf{x}, c)$.
$\mathfrak{m}H \cap I = \mathfrak{m}I$ and $I \subseteq H$ a minimal reduction so $\mathfrak{m}H \cap I = \mathfrak{m}I$.

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**Theorem 63.** *Let $(R, \mathfrak{m})$ be local and $I$ $\mathfrak{m}$-primary such that there exists an $i$ so that*

(a) $H_i(I) \neq 0$
(b) $\mu(I) \geq d + i$.

*If there exists $c \in annH_i(I) \setminus I$, then one of the following holds:*

(1) $(I : c) = (I\mathfrak{m} : c)$
(2) *there exists $J \subsetneq I$ and $x \in R$ so that $I = J + cx$ and $\mu(I) = \mu(J) + 1$ and $cH_i(J) = 0$ and $cH_{i-1}(J) = 0$.*

*Proof.* Suppose that (1) is not true. Then $(\mathfrak{m}I : c) \subsetneq (I : c)$ and there exists $x \in (I : c)$ so that $cx \notin \mathfrak{m}I$. Write $I = J + cx$. Then consider the following short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{H_i(J)}{cH_i(J)} & \longrightarrow & H_i(J,c) & \longrightarrow & \text{ann}_c H_{i-1}(J) & \longrightarrow & 0 \\ & & \wedge\vert & & \Vert & & \wedge\vert & & \\ 0 & \longrightarrow & \frac{H_i(J)}{cxH_i(J)} & \longrightarrow & H_i(J,cx) & \longrightarrow & \text{ann}_{cx} H_{i-1}(J) & \longrightarrow & 0. \end{array}$$

Thus, we see that $\lambda(H_i(J,cx)) = \lambda(H_i(I)) = \lambda(\operatorname{ann}_c H_i(I)) \underset{\text{lemma}}{=} \lambda(H_i(J,c))$. Therefore, $\lambda\left(\frac{H_i(J)}{cH_i(J)}\right) = \lambda\left(\frac{H_i(J)}{cxH_i(J)}\right)$, but $cxH_i(J) \subseteq cH_i(J)$ and by length we have $cxH_i(J) = cH_i(J)$. But we know that $x \in \mathfrak{m}$, which means that $cH_i(J) = 0$ by NAK. THus, $\operatorname{ann}_c H_{i-1}(J) = \operatorname{ann}_{cx} H_{i-1}(J)$.

$$0 \longrightarrow \operatorname{ann}_c H_{i-1}(J) \longrightarrow H_{i-1}(J) \xrightarrow{\cdot c} H_{i-1}(J) \longrightarrow \frac{H_{i-1}(J)}{cH_{i-1}(J)} \longrightarrow 0$$

$$0 \longrightarrow \operatorname{ann}_{cx} H_{i-1}(J) \longrightarrow H_{i-1}(J) \xrightarrow{\cdot cx} H_{i-1}(J) \longrightarrow \frac{H_{i-1}(J)}{cxH_{i-1}(J)} \longrightarrow 0.$$

**Claim 63.1.** *$J$ is $\mathfrak{m}$-primary.*

*Proof.* Assume not. Then there exists $Q \in \operatorname{Spec}(R)$ so that $\operatorname{ht}(Q) \leq d-1$ and $J \subseteq Q$ and $c \notin Q$. Indeed, if $c \in Q$, then $(J,c) \subseteq Q$ and so $I = (J,cx) \subseteq Q$, which would contradict the fact that $I$ is $\mathfrak{m}$-primary.

Let $\mu(I) = r+1$. Then $I = (z_1,\ldots,z_r,cx)$ and $J = (z_1,\ldots,z_r)$. Note that $cH_i(J) = cH_i(z_1,\ldots,z_r;R) = 0$. Now localize at $Q$ and note that $cH_i(z_1,\ldots,z_r;R_Q) = 0$ but $c$ is a unit in $R_Q$ and so we see that $H_i(z_1,\ldots,z_r;R_Q) = 0$. Thus, $r - i < \operatorname{grade}(JR_Q) \leq \operatorname{ht}(JR_Q) \leq d-1$. Thus, $d+i \leq \mu(I) = r+1 < d+i$, which is a contradiction. ☐

☐

**Corollary 64.** *If $I$ is $\mathfrak{m}$-primary and $c \in \operatorname{ann} H_1(I) \setminus I$ and $\mu(I) \geq d+1$, then $(I:c) = (\mathfrak{m}I : c)$.*

*Proof.* There exists $J \subseteq I$ so that $J + cx = I$. Then $cH_1(J) = cH_0(J) = 0$ and $H_0(J) = (R)/(J)$, which means that $c \in J$. This contradicts the fact that $\mu(I) = \mu(J)+1$. ☐

**Corollary 65.** *Suppose that $I$ is $\mathfrak{m}$-primary and $\mu(I) \geq d+1$. If $c \in R \setminus I$ and $cH_1(I)$ and $c \in (I:\mathfrak{m})$, then $c \in \overline{I}$.*

*Proof.* If $c \in (I:\mathfrak{m})$ then $c\mathfrak{m} \subseteq I$ and so $\mathfrak{m} \subseteq (I:c) = (\mathfrak{m}I : c)$, which means that $c\mathfrak{m} \subseteq I\mathfrak{m}$. By the determinant trick we see that $c \in \overline{I}$. ☐

**Corollary 66.** *Suppose that $I$ is $\mathfrak{m}$-primary and $H_1(I) \neq 0$ and $I = \overline{I}$. Then $\operatorname{ann} H_1(I) = I$ (and is contained in $\overline{I}$).*

*Proof.*

**Claim 66.1.** $\mu(I) \geq d+1$.

Assume the claim and $I \subsetneq \operatorname{ann} H_1(I)$. Then we see that there exists $c \in (\operatorname{ann}(H_1(I)) \setminus I) \cap (I:\mathfrak{m})$. Now go modulo $I$ and we see that $\operatorname{soc}(R)/(I) \cap (JR)/(I) \neq 0$ for all $I \subset J \subset R$ and so $c \in \overline{I} = I$ by the previous corollary. ☐

3/9/07

**Theorem 67.** *Let $(S,\mathfrak{m})$ be a local ring and set $R = (S)/(I)$ for $I \subseteq S$. Let $J = (I :_S \mathfrak{m})$. If $\overline{I} \neq \overline{J}$, then $b_{i+1}(M) - b_i(M) \geq 0$ for all $i \geq 1$ and for all $M$ finitely generated.*

*Proof.*

$$\begin{array}{ccccc} & & i^{\text{th}} & & (i-1)^{\text{st}} \\ 0 \longrightarrow & K \longrightarrow & R^n & \longrightarrow & R^q \end{array}$$

a part of a minimal resolution of $M$, then $b_{i+1}(M) = \mu_R(K)$. Then $J\mathfrak{m} \subseteq I$ and so $\partial_i((JR)^n) \subseteq \mathfrak{m}JR^n \subseteq 0$. In particular, $JR^n \subseteq K \subseteq \mathfrak{m}R^n$. Let $\pi : S^n \longrightarrow R^n$ and $A = \pi^{-1}(K)$. Then $K = \frac{A}{IS^n}$ and so $JS^n \subseteq A \subseteq \mathfrak{m}S^n$. Thus, $\mathfrak{m}K = \frac{\mathfrak{m}A + IS^n}{IS^n}$ and so $\frac{K}{\mathfrak{m}K} = \frac{A}{\mathfrak{m}A + IS^n}$.

**Claim 67.1.** *$A = B + C$, where $C \subset IS^n$ and $\mu(B) + \mu(C) = \mu(A)$ and $\mu(C) = \dim_{\frac{S}{\mathfrak{m}}} \frac{\mathfrak{m}A + IS^n}{\mathfrak{m}A}$.*

*Proof.* ☐

Let $a \in J \setminus \overline{I}$. This means that $J \subseteq \overline{I}$ because $\overline{J} \subseteq \overline{\overline{I}} = \overline{I} \subseteq \overline{J} \Longrightarrow \overline{I} = \overline{J}$. Set $N = \frac{S^n}{B}$ and notice that $IN = \frac{IS^n+B}{B} = \frac{A}{B}$ and $JN = \frac{JS^n+B}{B} \subseteq \frac{A}{B}$ and $\frac{A}{B} = \frac{IS^n+B}{B} \subseteq \frac{JS^n+B}{B}$. Thus, we know that $IN = JN$. Then $\det(n_1, \dots, n_e) = N$ and

$$\begin{aligned} an_1 &= a_{11}n_1 + \cdots + a_{1e}n_e \\ &\vdots \qquad \vdots \\ an_e &= a_{1e}n_1 + \cdots + a_{1e}n_e, \end{aligned}$$

for $a_{i,j} \in I$. By the same argument of the determinant trick, there exists and equation

$$x = a^n + b_1 a^{n-1} + \cdots + b_n,$$

for $b_i \in I^i$. Let $x = \det(a_{i,j} - aI)$. Note that $x \neq 0$ because otherwise $a \in \overline{I}$, which would be a contradiction. We also know that $x^m \neq 0$ for all $m$ because otherwise $a \in \overline{I}$. Also, $xN = 0$ and $xS^n \subseteq B \subseteq \mathfrak{m}S^n$. Let $U = \{1, x, x^2, \dots\}$ and set $T = U^{-1}S$. Because of flatness of localization, we get

$$T^n = xS^n \otimes_S T \subseteq B \otimes_S T \subseteq \mathfrak{m}S^n \otimes_S T = T^n.$$

Therefore, $B \otimes_S T = T^n$ and so $\mu_S(B) \geq \mu_T(B \otimes_S T) = \mu_T(T^n) = n = b_i(M)$. Therefore, $B_{i+1}(M) \geq b_i(M)$. □

**Exercise 20** (5 pts.)**.** Let $R = \frac{S}{I}$ be such that $\operatorname{depth}(R) = 0$ and $I = \overline{I}$. Show that $b_{i+1}(M) \geq b_i(M)$ for all $i \geq 1$.

**Exercise 21** (10 pts.)**.** Let $R = \frac{S}{I}$ and let $J = (I :_S \mathfrak{m})$. Assume that $o(I) > o(J)$. Then $b_{i+1}(M) \geq b_i(M)$ for all $i \geq 1$.

Notice that the order is

$$o(I) = \max\{n : I \subseteq \mathfrak{m}^n\}.$$

**Remark 13.** $b^R_{i+1}(M) \geq b^R_1(M)$ and $b^R_{i+1}(M) > b^R_i(M)$. There is a generalization of the previous theorem. If $\overline{\frac{I+P}{P}} \neq \overline{\frac{J+P}{P}}$ for all $P \in \operatorname{Spec}(R)$ such that $\operatorname{grade}(P) \geq 1$ and $\operatorname{ht}(P) \leq k$, then $b_{i+1}(M) - b_i(M) \geq k$.

3/19/07

**Definition 19.** Let $k$ be a field and $G$ an additive abelian group that is totally ordered. A *$k$-valuation* $v : k^* \longrightarrow G$ (recall that $k^* = k \setminus \{0\}$) is a group homomorphism such that

$$v(x+y) \geq \min\{v(x), v(y)\},$$

$$v(xy) = v(x) + v(y) \tag{0}$$

$$v(1) = 0 \tag{1}$$

$$v(x^{-1}) = -v(x). \tag{2}$$

Usually, we have a group homomorphism $v : R \longrightarrow G$ for $R$ a domain so that $v(xy) = v(x) + v(y)$ and $v(x+y) \geq \min\{v(x), v(y)\}$ and we extend this to a $k$-valuation (where $k = \mathbb{Q}(R)$) by

$$v\left(\frac{x}{y}\right) = v(x) - v(y).$$

**Definition 20.** For $v$ a $k$-valuation, we call $\Gamma_v(k) = v(k) \subset G$ the *value group.*

**Definition 21.** Let $k$ be a field and $V \subset k$. Then $V$ is a *valuation domain* if $\mathbb{Q}(V) = k$ and for every $x \in k^*$, either $x \in V$ or $x^{-1} \in V$.

**Remark 14.** (1) Let $I, J \subseteq V$ for $V$ a valuation domain. Then either $I \subseteq J$ or $J \subseteq I$.

*Proof.* Assume that $I \not\subseteq J$. Then there exists $x \in I \setminus J$. Let $y \in J$. Then $\frac{x}{y} \in V$ or $\frac{y}{x} \in V$. If $\frac{x}{y} \in V$, then $x = y \cdot \frac{x}{y} \in J$, which is a contradiction. Hence, $\frac{y}{x} \in V$ and $y = x \cdot \frac{y}{x} \in I$ and so $J \subseteq I$. □

(2) There exists a unique maximal ideal in $V$. This is just

$$\mathfrak{m}_V = \{x \in V \setminus \{0\} \text{ so that } x^{-1} \notin V\} \cup \{0\}.$$

(3) If $I$ is a finitely generated ideal of $V$, then $I$ is principal.

Given $v : k^* \longrightarrow G$, let $R_v = \{x \in k^* : v(x) \geq 0\} \cup \{0\}$. It is enough to show that $R_v$ is closed under addition and multiplication to see that this is a subring. Since $R_v \subseteq k$, we see that $R_v$ is then a domain. Note that $0 = v(1) = v(x \cdot x^{-1}) = v(x) + v(x^{-1}) = v(x) - v(x)$ for any $x \neq 0$. Then $\mathfrak{m}_{R_v} = \{x : v(x) > 0\} \cup \{0\}$. Therefore, any valuation gives rise to a valuation domain.

Suppose now that we have a valuation domain $V$. We will show that this gives rise to a valuation. Let $k = \mathbb{Q}(V)$. Then $V^* \subset k^*$. Set $\gamma_V = \frac{k^*}{V^*}$. Let $v : k^* \longrightarrow \frac{k^*}{V^*} = \Gamma_V$ be the canonical quotient map. We need only check that $\Gamma_V$ to be a totally ordered group. Let $\alpha, \beta \in \Gamma_V$ and let $x, y \in k^*$ be such that $x + V^* = \alpha$ and $y + V^* = \beta$. We define our order by $\alpha \leq_{\Gamma_V} \beta$ if $\frac{y}{x} \in V$.

**Exercise 22.** $\leq_{\Gamma_V}$ is a total order.

Pick $x, y \in k^*$. We want to show that $v(x+y) \geq \min\{v(x), v(y)\}$ with the order $\leq_{\Gamma_V}$. Either $xy^{-1} \in V$ or $yx^{-1} \in V$. If $xy^{-1} \in V$, then $\frac{x+y}{y} \in V$ and so $v(x+y) \geq v(y) \geq \min\{v(x), v(y)\}$. On the other hand, reversing the roles of $x$ and $y$ gives us $yx^{-1} \in V \Longrightarrow v(x+y) \geq \min\{v(x), v(y)\}$.

**Lemma 68.** *Let $R$ be a domain and $k = \mathbb{Q}(R)$. Let $\mathfrak{m}$ be a prime ideal of $R$. For all $x \in k$, either $\mathfrak{m}R[x] \neq R[x]$ or $\mathfrak{m}R[x^{-1}] \neq R[x^{-1}]$.*

*Proof.* First localize at elements outside of $\mathfrak{m}$ and we have $(R_\mathfrak{m}, \mathfrak{m})$ is local. Thus, we may assume that $(R, \mathfrak{m})$ is local. Assume that $\mathfrak{m}R[x^{-1}] = R[x^{-1}]$. This means that $1 = a_0 + a_1x^{-1} + \cdots + a_nx^{-n}$, where $a_i \in \mathfrak{m}$. Then $(1 - a_0)x^n = a_1x^{n-1} + \cdots + a_n$. Note that $1 - a_0$ is a unit in $R$. Thus, $x^n = \widetilde{a_1}x^{n-1} + \cdots + \widetilde{a_n}$ and so $x \in \overline{R}$. Thus, $R \subseteq R[x]$ is an integral extension. Now by lying over of integral extensions, we see that $\mathfrak{m}R[x] \neq R[x]$. □